Inverse Limits of Spectral Triples

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Gelfand - Naǐmark theorem supplies a one to one correspondence between commutative C^*-algebras and locally compact Hausdorff spaces. So any noncommutative C^*-algebra can be regarded as a generalization of a topological space. Similarly a spectral triple is a generalization of a Riemannian manifold. An (infinitely listed) covering of a Riemannian manifold has natural structure of Riemannian manifold. Here we will consider the noncommutative generalization of this result.

Contents

1 Motivation. Preliminaries

Some notions of the geometry have noncommutative generalizations based from the Gelfand-Naǐmark theorem.

Theorem 1.1. *[1] (Gelfand-Naǐmark). Let A be a commutative C^*-algebra and let $\mathcal{X}$ be the spectrum of A. There is the natural $*$-isomorphism $\gamma : A \to C_0(\mathcal{X})$.*

From the theorem it follows that a (noncommutative) C^*-algebra may be regarded as a generalized (noncommutative) locally compact Hausdorff topological space.

Any Riemannian manifold M with a Spin structure defines the standard spectral triple $\left(M, L^2\left(M, S\right), \slashed{D}\right)$ [18], i.e. the spectral triple is pure algebraic construction of manifold with a Spin structure. If $\widetilde{M} \to M$ is an infinitely listed covering projection then there is a sequence of finitely listed covering projections

$$... \to M_2 \to M_1 \to M_0 = M$$

which induces natural covering projections $\pi_n : \widetilde{M} \to M_n$ and for any $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{M}$ from $\widetilde{x}_1 \neq \widetilde{x}_2$ it follows that there is $k \in \mathbb{N}$ which satisfies to the condition $\pi_k\left(\widetilde{x}_1\right) \neq \pi_k\left(\widetilde{x}_2\right)$. From the topology and the differential geometry one can construct properties of $\widetilde{M}$ from the above sequence of finite covering projections. We will develop a pure algebraic construction of $\widetilde{M}$. If we have a sequence of spectral triples $\left\{\left(C^\infty\left(M\right), L^2\left(M_n, S_n\right), \slashed{D}_n\right)\right\}_{n\in\mathbb{N}}$ then we will construct a triple $\left(C^\infty\left(\widetilde{M}\right), L^2\left(\widetilde{M}, \widetilde{S}\right), \widetilde{\slashed{D}}\right)$ which reflects properties of the Riemannian manifold $\widetilde{M}$ with its Spin structure. Since our construction is pure algebraic we can apply it to the noncommutative case. A noncommutative example of this construction is presented in the Section 7.

This article assumes elementary knowledge of following subjects:

1. Set theory [15],
2. Category theory [35],
3. General topology [27],
4. Algebraic topology including K-theory, [21,35],
5. Differential geometry [6,9,23],
6. C^*-algebras, C^*-Hilbert modules and K-theory [3,20,31].

The terms “set”, “family” and “collection” are synonyms.

Following table contains a list of special symbols.

Symbol	Meaning
A^+	Unitization of C^*- algebra A
A^0	Opposite algebra of A consisting of elements $\{a^0 : a \in A\}$ with product $a^0 b^0 = (ba)^0$.
$\widehat{A}$	Spectrum of C^* - algebra A with the hull-kernel topology (or Jacobson topology)
A^G	Algebra of G invariants, i.e. $A^G = \{a \in A \mid ga = a, \forall g \in G\}$
$\mathrm{Aut}(A)$	Group * - automorphisms of C^* algebra A
$B(H)$	Algebra of bounded operators on Hilbert space H
$\mathbb{C}$ (resp. $\mathbb{R}$)	Field of complex (resp. real) numbers
$C(\mathcal{X})$	C^*-algebra of continuous complex valued functions on a compact space $\mathcal{X}$
$C_0(\mathcal{X})$	C^*-algebra of continuous complex valued functions on a locally compact topological space $\mathcal{X}$ equal to 0 at infinity
$G(\widetilde{\mathcal{X}} \mid \mathcal{X})$	Group of covering transformations of a covering projection $\widetilde{\mathcal{X}} \to \mathcal{X}$ [35]
$\mathfrak{cl}(\mathcal{U}) \subset \mathcal{X}$	The closure of subset $\mathcal{U}$ of topological space $\mathcal{X}$.
δ_{ij}	Delta symbol. If $i = j$ then $\delta_{ij} = 1$. If $i \neq j$ then $\delta_{ij} = 0$
$\Gamma(\mathcal{X}, E)$	A $C(\mathcal{X})$-module of sections of a locally trivial vector bundle $E \in \mathrm{Vect}(\mathcal{X})$
H	Hilbert space
$\mathcal{H}_A$	Hilbert space over A (Definition 1.36)
$\mathcal{K} = \mathcal{K}(H)$	C^* - algebra of compact operators
$\mathcal{K}(X_A)$	C^*-algebra of compact operators of a Hilbert A module X_A
$K_i(A)$ $(i = 0, 1)$	K groups of C^*-algebra A
$\varprojlim$	Inverse limit
$M(A)$	A multiplier algebra of C^*-algebra A
$\mathbb{M}_n(A)$	The $n \times n$ matrix algebra over C^*- algebra A
$\mathbb{N}$	A set of positive integer numbers
$\mathbb{N}^0$	A set of nonnegative integer numbers
$\mathscr{P}(A)$	A category of finitely generated projective modules over A
S^n	The n-dimensional sphere
$SU(n)$	Special unitary group
TM (resp. T^*M)	Tangent (resp. cotangent) bundle of differentiable manifold M [23]
$U(H) \subset \mathcal{B}(H)$	Group of unitary operators on Hilbert space H
$U(A) \subset A$	Group of unitary operators of algebra A
$U(n) \subset GL(n, \mathbb{C})$	Unitary subgroup of general linear group
$\mathrm{Vect}(\mathcal{X})$	A category of locally trivial vector bundles over a topological space $\mathcal{X}$ [21]
$\mathbb{Z}$	Ring of integers
$\mathbb{Z}_n$	Ring of integers modulo n
$\overline{k} \in \mathbb{Z}_n$	An element in $\mathbb{Z}_n$ represented by $k \in \mathbb{Z}$
$X \backslash A$	Difference of sets $X \backslash A = \{x \in X \mid x \notin A\}$
$\lvert X \rvert$	Cardinal number of the finite set
$f\rvert_{A'}$	Restriction of a map $f : A \to B$ to $A' \subset A$, i.e. $f\rvert_{A'} : A' \to B$

Definition 1.2. Let

$$G_1 \leftarrow G_2 \leftarrow ... \tag{1}$$

be an infinite sequence of finite groups and epimorphisms, and let G be a group with epimorpisms $h_n : G \to G_n$. The sequence is said to be *coherent* if $\bigcap \ker h_n$ is trivial and the following diagram is commutative.

$$\begin{tikzcd} & G \arrow[dl, "h_n"'] \arrow[dr, "h_{n-1}"] & \\ G_n \arrow[rr] & & G_{n-1} \end{tikzcd}$$

A family $\{G^n \subset G\}_{n\in\mathbb{N}}$ is said to be a G-*covering* of the sequence (1) if following conditions hold:

1. If $m < n$ then $G^m \subset G^n$,
2. $|G^n| = |G_n|$ for any $n \in \mathbb{N}$,
3. $h_n(G^n) = G_n$,
4. $\bigcup_{n\in\mathbb{N}} G^n = G$.

Definition 1.3. Let us consider a sequence (1) groups and epimorphisms. There are an inverse limit $\overline{G} = \varprojlim G_n$ [35] and natural epimorphisms $h_n : \overline{G} \to G_n$. We say that element $\overline{g}$ *is represented* by the sequence $\{g_n \in G_n\}_{n\in\mathbb{N}}$ if $g_n = h_n(\overline{g})$. We will write $\overline{g} = \mathfrak{Rep}_G(\{g_n\})$.

Henceforth $\{x_\iota\}_{\iota\in I}$ means a set indexed by finite or countable set I of indexes.

Definition 1.4. [2] Any homomorphisms of rings $A \to B$ induces functor $\mathscr{P}(A) \to \mathscr{P}(B)$ of categories of finitely generated projective modules. The functor is given by $P \mapsto B \otimes_A P$ on objects, and $f \mapsto \mathrm{Id}_B \otimes f$ on morphisms. We call it the *extensions of scalars* functor.

1.1 Topology

1.1.1 Covering projections and partition of unity

Definition 1.5. [35] Let $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a continuous map. An open subset $\mathcal{U} \subset \mathcal{X}$ is said to be *evenly covered* by $\widetilde{\pi}$ if $\widetilde{\pi}^{-1}(\mathcal{U})$ is the disjoint union of open subsets of $\widetilde{\mathcal{X}}$ each of which is mapped homeomorphicaly onto $\mathcal{U}$ by $\widetilde{\pi}$. A continuous map $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X}$ is called a *covering projection* if each point $x \in \mathcal{X}$ has an open neighborhood evenly covered by $\widetilde{\pi}$. $\widetilde{\mathcal{X}}$ is called the *covering space* and $\mathcal{X}$ the *base space* of covering projection.

Definition 1.6. Let $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a covering projection. A connected open subset $\widetilde{\mathcal{U}} \subset \widetilde{\mathcal{X}}$ is said to be a *one-to-one subset* with respect to $\widetilde{\pi}$ if the restriction $\widetilde{\pi}_{\widetilde{\mathcal{U}}} : \widetilde{\mathcal{U}} \to \widetilde{\pi}\left(\widetilde{\mathcal{U}}\right)$ is a homeomorphism. The family $\left\{\widetilde{\mathcal{U}}_\iota\right\}_{\iota\in I}$ of one-to-one subsets with respect to $\widetilde{\pi}$ such that $\widetilde{\mathcal{X}} = \bigcap_{\iota\in I} \widetilde{\mathcal{U}}_\iota$ is said to be a *one-to-one covering* with respect to $\widetilde{\pi}$.

Definition 1.7. [35] A fibration $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ with unique path lifting is said to be *regular* if, given any closed path ω in $\mathcal{X}$, either every lifting of ω is closed or none is closed.

Definition 1.8. [35] Let $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a covering projection. A self-equivalence is a homeomorphism $f : \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}$ such that $p \circ f = p$). We denote this group by $G(\widetilde{\mathcal{X}}|\mathcal{X})$. This group is said to be the *group of covering transformations* of p or the *covering group*.

Proposition 1.9. *[35] If $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ is a regular covering projection and $\widetilde{\mathcal{X}}$ is connected and locally path connected, then $\mathcal{X}$ is homeomorphic to space of orbits of $G(\widetilde{\mathcal{X}}|\mathcal{X})$, i.e. $\mathcal{X} \approx \widetilde{\mathcal{X}}/G(\widetilde{\mathcal{X}}|\mathcal{X})$. So p is a principal bundle.*

In this article we consider second-countable locally compact Hausdorff spaces only [27]. So we will say a "topological space" (resp. "compact space") instead "locally compact second-countable Hausdorff space" (resp. "compact second-countable Hausdorff space").

Theorem 1.10. *[27] Every compact Hausdorff space is normal.*

Theorem 1.11. *[27]* ***Urysohn lemma.*** *Let $\mathcal{X}$ be a normal space, let $\mathcal{A}$, $\mathcal{B}$ be disjoint closed subsets of $\mathcal{X}$. Let $[a, b]$ be a closed interval in the real line. Then there exist a continuous map $f : \mathcal{X} \to [a, b]$ such that $f(\mathcal{A}) = \{a\}$ and $f(\mathcal{B}) = \{b\}$.*

Theorem 1.12. *[27]* ***Urysohn metrization theorem.*** *Every regular space with a countable basis is metrizable.*

From the Theorems 1.10 and 1.11 it follows that if $\mathcal{X}$ is locally compact Hausdorff space $x \in \mathcal{X}$, and $\mathcal{B}$ is closed subset of $\mathcal{X}$, such that $x \notin \mathcal{B}$ then there exist a continuous map $f : \mathcal{X} \to [a, b]$ such that $f(x) = a$ and $f(\mathcal{B}) = \{b\}$. It means that locally compact Hausdorff space is completely regular, whence $\mathcal{X}$ is regular (See [27]), and from the Theorem 1.12 it follows next corollary.

Corollary 1.13. *Every locally compact second-countable Hausdorff space is metrizable.*

Theorem 1.14. *[27] Every metrizable space is paracompact.*

Definition 1.15. [27] Let $\{\mathcal{U}_\alpha \in \mathcal{X}\}_{\alpha \in J}$ be an indexed open covering of $\mathcal{X}$. An indexed family of functions

$$\phi_\alpha : \mathcal{X} \to [0, 1]$$

is said to be a *partition of unity* , dominated by $\{\mathcal{U}_\alpha\}_{\alpha \in J}$, if:

1. $\phi_\alpha (\mathcal{X} \backslash \mathcal{U}_\alpha) = \{0\}$
2. The family $\{\text{Support}(\phi_\alpha) = \mathfrak{cl}(\{x \in \mathcal{X} \mid \phi_\alpha > 0\})\}$ is locally finite.
3. $\sum_{\alpha \in J} \phi_\alpha(x) = 1$ for any $x \in \mathcal{X}$.

Theorem 1.16. *[27] Let $\mathcal{X}$ be a paracompact Hausdorff space; let $\{\mathcal{U}_\alpha \in \mathcal{X}\}_{\alpha \in J}$ be an indexed open covering of $\mathcal{X}$. Then there exists a partition of unity, dominated by $\{\mathcal{U}_\alpha\}$.*

Theorem 1.17. *[30] Suppose X and Y are compact Hausdorff connected spaces and $p : Y \to X$ is a continuous surjection. If $C(Y)$ is a projective finitely generated Hilbert module over $C(X)$ with respect to the action*

$$(f\xi)(y) = f(y)\xi(p(y)),\ f \in C(Y), \xi \in C(X),$$

then p is a finite-fold (or a finilely listed) covering.

1.1.2 Vector bundles and projective modules

Definition 1.18. [21] Let $\mathcal{X}$ be a topological space. A *quasi-vector bundle with base $\mathcal{X}$* is given by

1. a finite dimensional $\mathbb{C}$-vector space E_x for any $x \in \mathcal{X}$,
2. a topology on the disjoint union $E = \bigsqcup E_x$

A quasi-vector bundle is denoted by $\xi = (E, \pi, \mathcal{X})$ or simply E if there is no risk of confusion. The space E is the *total space* of ξ and E_x is the *fiber* of ξ at the point x.

Remark 1.19. Above definition is a specific case of discussed in [21] definition which includes real vector bundles. This article discusses complex vector bundles only.

Definition 1.20. [21] Let $\xi = (E, \pi, \mathcal{X})$ and $\xi = (E', \pi', \mathcal{X}')$ be quasi-vector bundles. A *general morphism* from ξ to ξ' is given by a pair (f, g) of continuous maps such that

1. the diagram
$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{X}' \end{tikzcd}$$
is commutative.
2. The map $g_x : E_x \to E'_{f(x)}$ induced by g is $\mathbb{C}$-linear. General morphism can be composed by obvious way.

If ξ, ξ' have the same base $\mathcal{X} = \mathcal{X}'$, a *morphism* between ξ and ξ' is a general morphism (f, g) such that $f = \mathrm{Id}_{\mathcal{X}}$. Such a morphism will be simply called g in the sequel. The quasi-vector bundles with the same base $\mathcal{X}$ are objects of a category, whose arrows we have just defined.

1.21. [21] Let $\xi = (E, \pi, \mathcal{X})$ be a quasi-vector bundle, and let $\mathcal{X}'$ be a subspace of $\mathcal{X}$. The triple $\left(\pi^{-1}(\mathcal{X}'), \pi|_{\pi^{-1}(\mathcal{X}')}, \mathcal{X}'\right)$ defines a quasi-vector bundle ξ' which is called a restriction of ξ to $\mathcal{X}'$. We denote it by $\xi|_{\mathcal{X}'}$, $E|_{\mathcal{X}'}$ or even simply $E_{\mathcal{X}'}$.

Definition 1.22. [21] More generally, let $f : \mathcal{X}' \to \mathcal{X}$ be any continuous map. For any $x' \in \mathcal{X}'$, let $E'_{x'} = E_{f(x')}$. Then the set $E' = \bigsqcup_{x' \in \mathcal{X}'} E'_{x'}$ may be identified with the *fiber product* $\mathcal{X}' \times_{\mathcal{X}} E$, i.e. with the subset of $\mathcal{X}' \times E$ formed by pairs (x', e) such that $f(x') = \pi(e)$. If $\pi' : E' \to X'$ is defined by $\pi'(x', e) = x'$, then it is clear that the triple $(E', \pi', \mathcal{X}')$ defines a quasi-vector bundle over $\mathcal{X}'$, when we provide E' with the topology induced by $\mathcal{X}' \times E$. We write ξ' as $f^*(\xi)$, or $f^*(E)$: this is the *inverse image* (or the *pullback*) of ξ by f. We have $f^*(\xi) = \xi$ for $f = \mathrm{Id}_{\mathcal{X}}$, and also $(f \circ f')^* (\xi) = f'^* (f^* (\xi))$ if $f' : \mathcal{X}'' \to \mathcal{X}'$ is another continuous map. We also say the E' is the pullback of E by f.

1.23. [21] Let us now consider two quasi-vector bundles over $\mathcal{X}$ and a morphism $\alpha : E \to F$. If we let $E' = f^*(E)$ as in 1.22 and $F' = f^*(F')$ we can also define a morphism $\alpha' = f^*(\alpha)$ from E' to F' by the formula $\alpha'_{x'} = \alpha_{f(x')}$. If we identify E' with $\mathcal{X}' \times_{\mathcal{X}} E$ and F' with $\mathcal{X}' \times_{\mathcal{X}} F$, then α' is identified with $\mathrm{Id}_{\mathcal{X}'} \times_{\mathcal{X}} \alpha$ which proves the continuity of the map α'.

$$\begin{tikzcd}
 & f^*(F) \arrow[rr] \arrow[dd] & & F \arrow[dd] \\
f^*(E) \arrow[ur, "f^*(\alpha)"] \arrow[rr] \arrow[dr] & & E \arrow[ur, "\alpha"] \arrow[dr] & \\
 & \mathcal{X}' \arrow[rr] & & \mathcal{X}
\end{tikzcd}$$

Proposition 1.24. *[21] Let $f : \mathcal{X}' \to \mathcal{X}$ be a continuous map. Then the correspondence $E \mapsto f^*(E)$ and $\alpha \mapsto f^*(\alpha)$ induces a functor between the category of quasi-vector bundles over $\mathcal{X}$ and the category of quasi-vector bundles over $\mathcal{X}'$*

1.25. [21] Let $V = \mathbb{C}^n$ a finite dimensional vector space. There is a natural structure of quasi-vector bundle on the product $\mathcal{X} \times V$. Such bundles are called *trivial vector bundles*.

Definition 1.26. [21] Let $\xi = (E, \pi, \mathcal{X})$ be a quasi-vector bundle. Then ξ is said to be *"locally trivial"* or a *"vector bundle"* if for any $x \in \mathcal{X}$ there exists a neighborhood $\mathcal{U}$ such that restriction $\xi|_{\mathcal{U}}$ is isomorphic to a trivial bundle.

Fact 1.27. Vector bundles are in fact the objects of a full subcategory of the category of quasi-vector bundles considered in the Definition 1.20. We will denote this category by $\mathscr{E}(\mathcal{X})$. If $f : \mathcal{X}' \to \mathcal{X}$ is a continuous map then functor f^* defined in 1.22 defines a functor from $\mathscr{E}(\mathcal{X})$ to $\mathscr{E}(\mathcal{X}')$, because an inverse image of any vector bundle is also a vector bundle (See [21]).

1.28. [21] Let $A = C(\mathcal{X})$ be a ring of continuous complex valued functions on a compact space $\mathcal{X}$. If E is a vector bundle with base $\mathcal{X}$ then the set $\Gamma(\mathcal{X}, E)$ of continuous sections is an A-module under the operation $(\lambda \cdot s)(x) = \lambda(x) s(x)$ where $\lambda \in A$, $s \in \Gamma(\mathcal{X}, E)$.

Theorem 1.29. *[21] Serre - Swan theorem. Let $A = C(\mathcal{X})$ be a ring of continuous complex valued functions on a compact space $\mathcal{X}$. Then the section functor Γ induces an equivalence of categories $\mathscr{E}(\mathcal{X}) \approx \mathscr{P}(A)$, where $\mathscr{P}(A)$ is a category of finitely generated projective A-modules.*

1.30. Let $f : \mathcal{X}' \to \mathcal{X}$ a continuous map, there is an inverse image functor f^* from $\mathscr{E}(\mathcal{X}) \to \mathscr{E}(\mathcal{X}')$ described in 1.23. If we identify $\mathscr{E}(\mathcal{X})$ (resp. $\mathscr{E}(\mathcal{X}')$) with $\mathscr{P}(C(\mathcal{X}))$ (resp. $\mathscr{P}(C(\mathcal{X}'))$) then f^* may be interpreted as the "extensions of scalars" functor $\mathscr{P}(C(\mathcal{X})) \to \mathscr{P}(C(\mathcal{X}'))$ defined by $P \mapsto C(\mathcal{X}') \otimes P$ and $h \mapsto 1_{C(\mathcal{X}')} \otimes h$ for any morphism h in $\mathscr{P}(C(\mathcal{X}))$.

Definition 1.31. Let $\xi = (E, \pi, \mathcal{X})$ be a vector bundle, and let $\{\mathcal{U}_\iota \subset \mathcal{X}\}_{\iota \in I}$ be a family of open subsets such that $\mathcal{X} = \bigcap_{\iota \in I} \mathcal{U}_\iota$. If $\{s_\iota \in \Gamma(\mathcal{U}_\iota, E|_{\mathcal{U}_\iota})\}_{\iota \in I}$ is a family of sections such that for any $\iota', \iota'' \in I$ following condition hold

$$s_{\iota'}|_{\mathcal{U}_{\iota'} \bigcap \mathcal{U}_{\iota''}} = s_{\iota''}|_{\mathcal{U}_{\iota'} \bigcap \mathcal{U}_{\iota''}}$$

then there is the unique section $s \in \Gamma(\mathcal{X}, E)$ such that

$$s|_{\mathcal{U}_\iota} = s_\iota; \ \forall \iota \in I.$$

The section s is said to be the *gluing* of $\{s_\iota\}_{\iota \in I}$ and we will write $s = \mathfrak{Gluing}\left(\{s_\iota\}_{\iota \in I}\right)$.

Remark 1.32. The Definition 1.31 describes the gluing of sections of the sheaf (See [16] for details).

Definition 1.33. Let $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a covering projection and let $\widetilde{\mathcal{U}} \subset \widetilde{\mathcal{X}}$ be a one-to-one subset. Let $\xi = (E, \pi, \mathcal{X})$ be a vector bundle and let $\widetilde{\xi} = \left(\widetilde{E}, \pi', \widetilde{\mathcal{X}}\right)$ be a pullback of ξ. There is a natural isomorphism

$$\varphi : \Gamma\left(\widetilde{\pi}\left(\widetilde{\mathcal{U}}\right), E|_{\widetilde{\pi}(\widetilde{\mathcal{U}})}\right) \xrightarrow{\approx} \Gamma\left(\widetilde{\mathcal{U}}, \widetilde{E}|_{\widetilde{\mathcal{U}}}\right).$$

If $s \in \Gamma\left(\widetilde{\pi}\left(\widetilde{\mathcal{U}}\right), E|_{\widetilde{\pi}(\widetilde{\mathcal{U}})}\right)$ is a section then the section $\widetilde{s} = \varphi(s) \in \Gamma\left(\widetilde{\mathcal{U}}, \widetilde{E}|_{\widetilde{\mathcal{U}}}\right)$ is said to be the *$\widetilde{\mathcal{U}}$-pullback* of s and we will write $\widetilde{s} = \mathfrak{pullback}_{\widetilde{\mathcal{U}}}(s)$. Elements of $C_0(\mathcal{X})$ and $C_0\left(\widetilde{\mathcal{X}}\right)$ can be regarded as sections of one dimensional trivial bundles, so we will use notation $\widetilde{f} = \mathfrak{pullback}_{\widetilde{\mathcal{U}}}(f)$ where $f \in C_0(\mathcal{X})$ and $\widetilde{f} \in C_0\left(\widetilde{\pi}\left(\widetilde{\mathcal{U}}\right)\right)$.

Definition 1.34. Let us consider situation of the Definition 1.33. If be $s \in \Gamma(\mathcal{X}, E)$ is a section then following condition hold

$$\mathfrak{pullback}_{\widetilde{\mathcal{U}}'}(s)|_{\widetilde{\mathcal{U}}' \bigcap \widetilde{\mathcal{U}}''} = \mathfrak{pullback}_{\widetilde{\mathcal{U}}''}(s)|_{\widetilde{\mathcal{U}}' \bigcap \widetilde{\mathcal{U}}''}$$

for any $\widetilde{\mathcal{U}}'$ (resp. $\widetilde{\mathcal{U}}''$) one-to-one subsets. Hence there is the unique section $\widetilde{s} = \Gamma\left(\widetilde{\mathcal{X}}, \widetilde{E}\right)$ such that

$$\widetilde{s}|_{\widetilde{\mathcal{U}}} = \mathfrak{pullback}_{\widetilde{\mathcal{U}}}(s)$$

for any one-to-one subset $\widetilde{\mathcal{U}}$. The section $\widetilde{s}$ is said to be the $\widetilde{\pi}$-*pullback* of s and we will write

$$\widetilde{s} = \mathfrak{pullback}_{\widetilde{\pi}}(s).$$

It is clear that following condition hold

$$\mathfrak{pullback}_{\widetilde{\pi}}(s) = 1_{C(\widetilde{\mathcal{X}})} \otimes_{C(\mathcal{X})} s.$$

1.2 Hilbert modules

We refer to [3, 20] for the definition of Hilbert C^*-modules, or simply Hilbert modules. Denote by X_A a right Hilbert A-module. The sesquilinear product on a Hilbert module X_A will be denoted by $\langle\cdot,\cdot\rangle_{X_A}$. For any $\xi, \zeta \in X_A$ let us define an A-endomorphism $\theta_{\xi,\zeta}$ given by $\theta_{\xi,\zeta}(\eta) = \xi\langle\zeta,\eta\rangle_{X_A}$ where $\eta \in X_A$. Operator $\theta_{\xi,\zeta}$ shall be denoted by $\xi\rangle\langle\zeta$. Norm completion of algebra generated by operators $\theta_{\xi,\zeta}$ is said to be an algebra of compact operators $\mathcal{K}(X_A)$. We suppose that there is a left action of $\mathcal{K}(X_A)$ on X_A which is A-linear, i.e. action of $\mathcal{K}(X_A)$ commutes with action of A.

Definition 1.35. [20] An *A-B-correspondence* X is a right Hilbert B-module together with a $*$-homomorphism $\phi_X\colon A \to \mathscr{L}(X)$. We will denote this by ${}_AX_B$.

Definition 1.36. [3] Let A be a C^*-algebra, let $\mathcal{H}_A$ be the completion of the direct sum of a countable number of copies of A, i.e. $\mathcal{H}_A$ consists of all sequences $\{a_n \in A\}_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} a_n^* a_n$ converges, with inner product

$$\langle\{a_n\},\{a_n\}\rangle_{\mathcal{H}_A} = \sum_{n=1}^{\infty} a_n^* b_n.$$

$\mathcal{H}_A$ is said to be the *Hilbert space over A*. Henceforth the Hilbert space over A will by denoted by $\mathcal{H}_A$.

The sesquilinear product on a Hilbert space H will be denoted by $(\cdot,\cdot)$. For any $\xi, \zeta \in H$ let us define an operator $\Theta_{\xi,\zeta} \in B(H)$ given by $\Theta_{\xi,\zeta}(\eta) = (\zeta,\eta)\xi$ where $\eta \in H$. Operator $\theta_{\xi,\zeta}$ shall be denoted by $\xi)(\zeta$.

1.3 Riemannian manifolds and covering projections

Proposition 1.37. *(Proposition 5.9 [23])*

1. *Given a connected manifold M there is a unique (unique up to isomorphism) universal covering manifold, which will be denoted by $\widetilde{M}$.*
2. *The universal covering manifold $\widetilde{M}$ is a principal fibre bundle over M with group $\pi_1(M)$ and projection $p : \widetilde{M} \to M$, where $\pi_1(M)$ is the first homotopy group of M.*

3. *The isomorphism classes of covering spaces over M are in 1:1 correspondence with the conjugate classes of subgroups of $\pi_1(M)$. The correspondence is given as follows. To each subgroup H of $\pi_1(M)$, we associate $E = \widetilde{M}/H$. Then the covering manifold E corresponding to H is a fibre bundle over M with fibre $\pi_1(M)/H$ associated with the principal bundle $\widetilde{M}(M, \pi_1(M))$. If H is a normal subgroup of $\pi_1(M)$, $E = \widetilde{M}/H$ is a principal fibre bundle with group $\pi_1(M)/H$ and is called a regular covering manifold of M.*

Proposition 1.38. *[6] A differential manifold M admits a (smooth) partition of unity if and only if it is paracompact.*

1.39. If $\widetilde{M}$ is a covering space of Riemannian manifold M then it is possible to give $\widetilde{M}$ a Riemannian structure such that $\pi : \widetilde{M} \to M$ is a local isometry (this metric is called the *covering metric*). See [9] for details.

1.4 Strong and/or weak extension

In this section I follow to [31].

Definition 1.40. [31] Let A be a C^*-algebra. The *strict topology* on $M(A)$ is the topology generated by seminorms $|||x|||_a = \|ax\| + \|xa\|$, ($a \in A$). If $x \in M(A)$ and a sequence of partial sums $\sum_{i=1}^{n} a_i$ ($n = 1, 2, ...$), ($a_i \in A$) tends to x in the strict topology then we shall write

$$x = \sum_{i=1}^{\infty} a_i.$$

Definition 1.41. [31] Let H be a Hilbert space. The *strong* topology on $B(H)$ is the locally convex vector space topology associated with the family of seminorms of the form $x \mapsto \|x\xi\|$, $x \in B(H)$, $\xi \in H$.

Definition 1.42. [31] Let H be a Hilbert space. The *weak* topology on $B(H)$ is the locally convex vector space topology associated with the family of seminorms of the form $x \mapsto |(x\xi, \eta)|$, $x \in B(H)$, $\xi, \eta \in H$.

Theorem 1.43. *[31] Let M be a C^*-subalgebra of $B(H)$, containing the identity operator. The following conditions are equivalent:*

1. *$M = M''$ where M'' is the bicommutant of M.*
2. *M is weakly closed.*
3. *M is strongly closed.*

Definition 1.44. Any C^*-algebra M is said to be a *von Neumann algebra* or a *W^*- algebra* if M satisfies to the conditions of the Theorem 1.43.

Definition 1.45. [31] Let $B \subset B(H)$ be a C^*-algebra and B acts non-degenerately on H. Denote by B'' the strong closure of B in $B(H)$. B'' is an unital weakly closed C^*-algebra . The algebra B'' is said to be the *bicommutant*, or the *enveloping von Neumann algebra* or the *enveloping W^*-algebra* of B.

1.46. Any separable C^*-algebra A has a state τ which induces a faithful *GNS* representation [28]. There is a $\mathbb{C}$-valued product on A given by

$$(a, b) = \tau\left(a^* b\right).$$

This product induces a product on $A/\mathcal{I}_\tau$ where $\mathcal{I}_\tau = \{a \in A \mid \tau(a^*a) = 0\}$. So $A/\mathcal{I}_\tau$ is a phe-Hilbert space. Let denote by $L^2\left(A, \tau\right)$ the Hilbert completion of $A/\mathcal{I}_\tau$. The Hilbert space $L^2\left(A, \tau\right)$ is a space of a faithful GNS representation of A.

1.47. If $\mathcal{X}$ is a second-countable locally compact Hausdorff space then $C_0\left(\mathcal{X}\right)$ is a separable algebra [7]. Therefore $C_0\left(\mathcal{X}\right)$ has a state τ such that associated with τ *GNS* representation [28] is faithful. From [4] it follows that the state τ can be represented as the following integral

$$\tau\left(a\right) = \int_{\mathcal{X}} a \; d\mu \tag{2}$$

where μ is a positive measure. In analogy with the Riemann integration, one can define the integral of a bounded continuous function a on $\mathcal{X}$. There is a $\mathbb{C}$ valued product on $C_0\left(\mathcal{X}\right)$ given by

$$(a, b) = \tau\left(a^* b\right) = \int_{\mathcal{X}} a^* b \; d\mu,$$

whence $C_0\left(\mathcal{X}\right)$ is a phe-Hilbert space. Denote by $L^2\left(C_0\left(\mathcal{X}\right), \tau\right)$ or $L^2\left(\mathcal{X}, \mu\right)$ the Hilbert space completion of $C_0\left(\mathcal{X}\right)$. From [28, 34] it follows that W^*-enveloping algebra $C_0\left(\mathcal{X}\right)$ is isomorphic to the algebra $L^\infty\left(\mathcal{X}, \mu\right)$ (of classes of) essentially bounded complex-valued measurable functions. The $L^\infty\left(\mathcal{X}, \mu\right)$ is a C^*-algebra with the pointwise-defined opreations and the essential norm $f \mapsto \|f\|_\infty$.

1.5 Nonstandard analysis

1.5.1 Riemannian integration

Nonstandard analysis operates with actual infinitesimally small parameters. This procedure enables us replace the Riemannian integration with the summation of infinitesimally small elements. Strict explanation of nonstandard analysis and its applications are contained in [19]. Here is an informal explanation. Suppose that $f \in C([0,1])$ is a continuous function and we would like to define the integral

$$\int_0^1 f\left(x\right) dx.$$

Let Q be a countable set given by $Q = \left\{x \in \mathbb{Q} \bigcap [0,1] \; \& \; \exists m, n \in \mathbb{N}^0 \;\; x = \frac{m}{2^n}\right\}$. Then the integral can be represented as a sum of infinitesimally small numbers

$$\int_0^1 f\left(x\right) dx = \sum_{q \in Q} a^q.$$

Indeed infinitesimally small number x is a sequence $\{x_n \in \mathbb{C}\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = 0$. Suppose that a_q is represented by the sequence $\{a_n^q \in \mathbb{C}\}_{n\in\mathbb{N}}$ such that

$$a_n^q = \begin{cases} 0 & \mathrm{den}(q) < 2^n \\ \frac{f(q)}{2^n} & \mathrm{den}(q) \geq 2^n \end{cases}$$

where den means denominator of the irreducible fraction, i.e. $\mathrm{den}\left(\frac{x}{y}\right) = y$. It is clear that

$$\int_0^1 f(x)\, dx = \lim_{n\to\infty} \sum_{q\in Q} a_n^q = \sum_{q\in Q} a^q$$

and

$$\lim_{n\to\infty} a_n^q = 0.$$

Above equations mean that the Riemannian integral can be represented as a sum of infinitesimally small numbers.

1.5.2 Application to infinitely listed covering projections

Let us consider the following sequence of covering projections

$$S^1 \xleftarrow{f} S^1 ... \xleftarrow{f} S^1 \xleftarrow{f} S^1 \xleftarrow{f} ... \xleftarrow{f} .... \leftarrow \mathbb{R}. \tag{3}$$

Roughly speaking there is a sequence of homomorphisms

$$C(S^1) \to C(S^1) \to ... \to C(S^1) \to C(S^1) \to ... \to .... \to C_0(\mathbb{R}).$$

Indeed there is no a natural homomorphism $C(S^1) \to C_0(\mathbb{R})$, there is a correspondence ${}_{C_0(\mathbb{R})}X_{C(S^1)}$. We would like represent functions in $C_0(\mathbb{R})$ by functions on S^1. If $\mathbb{R} \to S^1$ is a covering projection then any $f \in C\left(S^1\right)$ can be represented by a 2π periodic function $\widetilde{f} \in C_b(\mathbb{R})$ given by Fourier series

$$\widetilde{f}(\xi) = \sum_{m\in\mathbb{Z}} a_m e^{im\xi}.$$

If $\mathbb{R} \to \mathcal{X}_n \xrightarrow{n-\text{listed}} S^1$ then any $f_n \in C(\mathcal{X}_n)$ can be represented as a $2\pi n$ periodic function $\widetilde{f}_n$ given by

$$\widetilde{f}_n(\xi) = \sum_{m\in\mathbb{Z}} a_m e^{\frac{im\xi}{n}}.$$

All finite listed covering projections from the sequence give following functions

$$\widetilde{f}(\xi) = \sum_{q\in\mathbb{Q}} a_q e^{iq\xi}. \tag{4}$$

These functions cannot represent any nontrivial function in $C_0(\mathbb{R})$. But $C_0(\mathbb{R})$ can be represented by a Fourier transform

$$f(\xi) = \int_{\mathbb{R}} \widehat{f}(x) e^{-2\pi i x \xi}$$

where $\widehat{f} \in L^1(\mathbb{R})$. However the Fourier transform can be regarded as the series (4) with infinitesimally small coefficients. Let us consider a dependent on $n \in \mathbb{N}$ series

$$\sum_{q\in\mathbb{Q}} a_q^n e^{iq\xi}$$

such that

$$a_q^n \to 0; \sum_{q\in\mathbb{Q}} a_q^n e^{iq\xi} \to f(\xi); \text{ as } n \to \infty,$$

i.e. a_q^n can be regarded as infinitesimally small coefficients. Any $f \in C_0(\mathbb{R})$ is a weak (pointwise) limit of the sequence of periodic functions $\{f_n \in C_b(\mathbb{R})\}_{n\in\mathbb{N}}$ given by

$$f_n(\xi) = f(\xi) - (\xi - n\pi) \frac{f(n\pi) - f(-n\pi)}{2\pi n} - f(-n\pi), \ \forall \xi \in [-n\pi, n\pi];$$
$$f_n(\xi + 2\pi n) = f(\xi), \ \forall \xi \in \mathbb{R}.$$

From periodicity of f_n it follows that

$$f_n = \sum_{q\in\mathbb{Q}} a_q^n e^{2\pi i q \xi}$$

and it is clear that $a_q^n \to 0$ as $n \to \infty$ for any $q \in \mathbb{Q}$.

2 Spin manifolds and spectral triples

This section contains citations of [18].

2.1 Clifford algebras

2.1. *Clifford algebras*. We start with (V, g), where $V \cong \mathbb{R}^n$ and g is a *nondegenerate symmetric bilinear form*. If $q(v) = g(v, v)$, then $2g(u, v) = q(u + v) - q(u) - q(v)$. Thus g is determined by the corresponding "quadratic form" q.

Definition 2.2. [18] The *Clifford algebra* $\mathbb{C}\ell(V, g)$ is an algebra (over $\mathbb{R}$) generated by the vectors $v \in V$ subject to the relations $uv + vu = 2g(u, v)1$ for $u, v \in V$.

The existence of this algebra can be seen in two ways. First of all, let $\mathcal{T}(V)$ be the tensor algebra on V, that is, $\mathcal{T}(V) := \bigoplus_{k=0}^{\infty} V^{\otimes n}$. Then

$$\mathbb{C}\ell(V, g) := \mathcal{T}(V) / \operatorname{Ideal}\langle u \otimes v + v \otimes u - 2g(u, v)\, 1 : u, v \in V\rangle. \tag{5}$$

Since the relations are *not* homogeneous, the $\mathbb{Z}$-grading of $\mathcal{T}(V)$ is lost, only a $\mathbb{Z}_2$-grading remains:

$$\mathrm{C}\ell(V,g) = \mathrm{C}\ell^0(V,g) \oplus \mathrm{C}\ell^1(V,g).$$

The second option is to define $\mathrm{C}\ell(V,g)$ as a subalgebra of $\mathrm{End}_{\mathbb{R}}(\Lambda^\bullet V)$ generated by all expressions $c(v) = \varepsilon(v) + \iota(v)$ for $v \in V$, where

$$\varepsilon(v) \colon u_1 \wedge \cdots \wedge u_k \mapsto v \wedge u_1 \wedge \cdots \wedge u_k$$
$$\iota(v) \colon u_1 \wedge \cdots \wedge u_k \mapsto \sum_{j=1}^{k} (-1)^{j-1} g(v,u_j) u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_k.$$

Note that $\varepsilon(v)^2 = 0$, $\iota(v)^2 = 0$, and $\varepsilon(v)\iota(u) + \iota(u)\varepsilon(v) = g(v,u)\,1$. Thus

$$c(v)^2 = g(v,v)\,1 \quad \text{for all} \quad v \in V,$$
$$c(u)c(v) + c(v)c(u) = 2g(u,v)\,1 \quad \text{for all} \quad u,v \in V.$$

Thus these operators on $\Lambda^\bullet V$ do provide a representation of the algebra (5).
Dimension count: suppose $\{e_1, \dots, e_n\}$ is an orthonormal basis for (V,g), i.e., $g(e_k,e_k) = \pm 1$ and $g(e_j,e_k) = 0$ for $j \neq k$. Then the $c(e_j)$ anticommute and thus a basis for $\mathrm{C}\ell(V,g)$ is $\{c(e_{k_1}) \dots c(e_{k_r}) : 1 \leq k_1 < \cdots < k_r \leq n\}$, labelled by $K = \{k_1, \dots, k_r\} \subseteq \{1, \dots, n\}$. Indeed,

$$c(e_{k_1}) \dots c(e_{k_r}) \colon 1 \mapsto e_{k_1} \wedge \cdots \wedge e_{k_r} \equiv e_K \in \Lambda^\bullet V$$

and these are linearly independent. Thus the dimension of the subalgebra of $\mathrm{End}_{\mathbb{R}}(\Lambda^\bullet V)$ generated by all $c(v)$ is just $\dim \Lambda^\bullet V = 2^n$. Now, a moment's thought shows that in the abstract presentation (5), the algebra $\mathrm{C}\ell(V,g)$ is generated as a vector space by the 2^n products $e_{k_1} e_{k_2} \dots e_{k_r}$, and these are linearly independent since the operators $c(e_{k_1}) \dots c(e_{k_r})$ are linearly independent in $\mathrm{End}_{\mathbb{R}}(\Lambda^\bullet V)$. Therefore, this representation of $\mathrm{C}\ell(V,g)$ is faithful, and $\dim \mathrm{C}\ell(V,g) = 2^n$.
The so-called "symbol map":

$$\sigma : a \mapsto a(1) : \mathrm{C}\ell(V,g) \to \Lambda^\bullet V$$

is inverted by a "quantization map":

$$Q : u_1 \wedge u_2 \wedge \cdots \wedge u_r \longmapsto \frac{1}{r!} \sum_{\tau \in S_r} (-1)^\tau \, c(u_{\tau(1)}) c(u_{\tau(2)}) \dots c(u_{\tau(r)}). \tag{6}$$

To see that it is an inverse to σ, one only needs to check it on the products of elements of an orthonormal basis of (V,g). From now, we write uv instead of $c(u)c(v)$, etc., in $\mathrm{C}\ell(V,g)$.

Proposition 2.3. *[18] There is an unique trace $\tau : \mathrm{C}\ell(V,g) \to \mathbb{C}$ such that $\tau(1) = 1$ and $\tau(a) = 0$ for a odd.*

There is a useful universality property of Clifford algebras, which is an immediate consequence of their definition.

Lemma 2.4. *[18] Any $\mathbb{R}$-linear map $f\colon V \to A$ (an $\mathbb{R}$-algebra) that satisfies*

$$f(v)^2 = g(v,v)\,1_A \quad \textit{for all} \quad v \in V$$

extends to an unique unital $\mathbb{R}$-algebra homomorphism $\tilde{f}\colon \mathrm{C}\ell(V,g) \to A$.

Here are a few applications of universality that yield several useful operations on the Clifford algebra.

1. *Grading*: take $A = \mathrm{C}\ell(V,g)$ itself; the linear map $v \mapsto -v$ on V extends to an *automorphism* $\chi \in \mathrm{Aut}(\mathrm{C}\ell(V,g))$ satisfying $\chi^2 = \mathrm{id}_A$, given by

   $$\chi(v_1 \dots v_r) := (-1)^r\, v_1 \dots v_r.$$

   This operator gives the $\mathbb{Z}_2$-grading

   $$\mathrm{C}\ell(V,g) =: \mathrm{C}\ell^0(V,g) \oplus \mathrm{C}\ell^1(V,g).$$

2. *Reversal*: take $A = \mathrm{C}\ell(V,g)^{\mathrm{op}}$, the opposite algebra. Then the map $v \mapsto v$, considered as the inclusion $V \hookrightarrow A$, extends to an *antiautomorphism* $a \mapsto a^!$ of $\mathrm{C}\ell(V,g)$, given by $(v_1 v_2 \dots v_r)^! := v_r \dots v_2 v_1$.

3. *Complex conjugation*: the *complexification* of $\mathrm{C}\ell(V,g)$ is $\mathrm{C}\ell(V,g) \otimes_{\mathbb{R}} \mathbb{C}$, which is isomorphic to $\mathrm{C}\ell(V^{\mathbb{C}}, g^{\mathbb{C}})$ as a $\mathbb{C}$-algebra. Now take A to be $\mathrm{C}\ell(V,g) \otimes_{\mathbb{R}} \mathbb{C}$ and define $f\colon v \mapsto \bar{v} : V^{\mathbb{C}} \to V^{\mathbb{C}} \hookrightarrow A$ (a real-linear map). It extends to an *antilinear automorphism* of A.

4. *Adjoint*: Also, $a^* := (\bar{a})^!$ is an *antilinear involution* on $\mathrm{C}\ell(V,g) \otimes_{\mathbb{R}} \mathbb{C}$.

5. *Charge conjugation*: $\kappa(a) := \chi(\bar{a}) : v_1 \dots v_r \mapsto (-1)^r \bar{v}_1 \dots \bar{v}_r$ is an antilinear automorphism of $\mathrm{C}\ell(V,g) \otimes_{\mathbb{R}} \mathbb{C}$.

From now on, $n = 2m$ for n even, $n = 2m+1$ for n odd. We take $\mathbb{C}\mathrm{l}(V) \cong \mathrm{C}\ell(V,g) \otimes_{\mathbb{R}} \mathbb{C}$ with g always positive definite. Suppose $\{e_1, \dots, e_n\}$ is an *oriented* orthonormal basis for (V,g). If $e'_k = \sum_{j=1}^n h_{jk}\, e_j$ with $H^t H = 1_n$, then $e'_1 \dots e'_n = (\det H)\, e_1 \dots e_n$, and $\det H = \pm 1$. We restrict to the oriented case $\det H = +1$, so the expression $e_1 e_2 \dots e_n$ is independent of $\{e_1, e_2, \dots, e_n\}$. Thus

$$\gamma := (-i)^m e_1 e_2 \dots e_n$$

is well-defined in $\mathbb{C}\mathrm{l}(V)$. Now

$$\gamma^* = i^m e_n \dots e_2 e_1 = (-i)^m (-1)^m (-1)^{n(n-1)/2} e_1 e_2 \dots e_n = (-1)^m (-1)^{n(n-1)/2} \gamma,$$

and

$$\frac{n(n-1)}{2} = \left\{ \begin{matrix} m(2m-1), & n \text{ even} \\ (2m+1)m, & n \text{ odd} \end{matrix} \right\} \equiv m \bmod 2,$$

so $\gamma^* = \gamma$. But also $\gamma^*\gamma = (e_n \dots e_2 e_1)(e_1 e_2 \dots e_n) = (+1)^n = 1$, so γ is "unitary". Hence $\gamma^2 = 1$, so $\frac{1+\gamma}{2}$, $\frac{1-\gamma}{2}$ are "orthogonal projectors" in $\mathbb{C}\mathrm{l}(V)$.

Since $\gamma e_j = (-1)^{n-1} e_j \gamma$, we get that if n is odd, then γ is *central* in $\mathbb{C}l(V)$; and for n even, γ anticommutes with V, but is central in the even subalgebra $C\ell^0(V)$. Moreover, when n is even and $v \in V$, then $\gamma v \gamma = -v$, so that $\gamma(\cdot)\gamma = \chi \in \operatorname{Aut}(\mathbb{C}l(V))$.

Proposition 2.5. *[18] The centre of* $\mathbb{C}l(V)$ *is* $\mathbb{C}1$ *if n is even; and it is* $\mathbb{C}1 \oplus \mathbb{C}\gamma$ *if n is odd.*

2.2 Clifford algebra bundles

A real vector bundle $E \to M$ is a *Euclidean bundle* if, with $\mathcal{E} = \Gamma(M, E^{\mathbb{C}})$, there is a symmetric A-bilinear form $g\colon \mathcal{E} \times \mathcal{E} \to A = C(M)$ such that

1. $g(s,t) \in C(M;\mathbb{R})$ when s, t lie in $\Gamma(M, E)$ —the real sections;
2. $g(s,s) \geq 0$ for $s \in \Gamma(M, E)$, with $g(s,s) = 0 \implies s = 0$.

By defining $(s \mid t) := g(s^*, t)$, we get a *hermitian pairing* with values in A:

- $(s \mid t)$ is A-linear in t;
- $(t \mid s) = \overline{(s \mid t)} \in A$;
- $(s \mid s) \geq 0$, with $(s \mid s) = 0 \implies s = 0$ in $\mathcal{E}$;
- $(s \mid ta) = (s \mid t)\, a$ for all $s, t \in \mathcal{E}$ and $a \in A$.

These properties make $\mathcal{E}$ a (right) C^*-module over A, with C^*-norm given by

$$\|s\|_{\mathcal{E}} := \sqrt{\|(s \mid s)\|_A} \quad \text{for } s \in \mathcal{E}.$$

For each $x \in M$, we can form $\mathbb{C}l(E_x) := C\ell(E_x, g_x) \otimes_{\mathbb{R}} \mathbb{C}$. Using the linear isomorphisms $\sigma_x\colon \mathbb{C}l(E_x) \to (\Lambda^\bullet E_x)^{\mathbb{C}}$, we see that these are fibres of a vector bundle $\mathbb{C}l(E) \to M$, isomorphic to $(\Lambda^\bullet E)^{\mathbb{C}} \to M$ as $\mathbb{C}$-vector bundles (but not as algebras!). Under $(\kappa\lambda)(x) := \kappa(x)\lambda(x)$, the sections of $\mathbb{C}l(E)$ also form an algebra $\Gamma(M, \mathbb{C}l(E))$. It has an A-valued pairing

$$(\kappa \mid \lambda)\colon x \mapsto \tau(\kappa(x)^*\lambda(x)).$$

By defining $\|\kappa\| := \sup_{x \in M} \|\kappa(x)\|_{\mathbb{C}l(E_x)}$, this becomes a C*-algebra.

Definition 2.6. [18] A *Clifford module* over (M, g) is a finitely generated projective A-module, with $A = C(M)$, of the form $\mathcal{E} = \Gamma(M, E)$ for E a (complexified) Euclidean bundle, together with an A-linear homomorphism $c\colon B \to \Gamma(M, \operatorname{End} E)$, where $B := \Gamma(M, \mathbb{C}l(T^*M))$ is the Clifford algebra bundle generated by $\mathcal{A}^1(M)$, such that

$$(s \mid c(\kappa)t) = (c(\kappa^*)s \mid t) \quad \text{for all} \quad s, t \in \mathcal{E},\ \kappa \in B.$$

2.3 Riemannian geometry

Let M be a *compact* C^∞ manifold *without boundary*, of dimension n. Compactness is not crucial for some of our arguments (although it may be for others), but is very convenient, since it means that the algebras $C(M)$ and $C^\infty(M)$ are *unital*: the unit is the constant function 1. For convenience we use the function algebra $A = C(M)$ —a commutative C^*-algebra— at the beginning. We will change to $\mathcal{A} = C^\infty(M)$ later, when the differential structure becomes important. Any A-module (or more precisely, a "symmetric A-bimodule") which is *finitely generated and projective* is of the form $\mathcal{E} = \Gamma(M, E)$ for $E \to M$ a (complex) vector bundle. Two important cases are

$$\begin{aligned}\mathfrak{X}(M) &= \Gamma(M, T_{\mathbb{C}}M) = \text{ (continuous) vector fields on } M;\\ \mathcal{A}^1(M) &= \Gamma(M, T^*_{\mathbb{C}}M) = \text{ (continuous) 1-forms on } M.\end{aligned}$$

These are *dual* to each other: $\mathcal{A}^1(M) \cong \mathrm{Hom}_A(\mathfrak{X}(M), A)$, where Hom_A means "A-module maps" commuting with the action of A (by multiplication).

Definition 2.7. [18] A *Riemannian metric* on M is a symmetric bilinear form

$$g \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to C(M)$$

such that:

1. $g(X, Y)$ is a real function if X, Y are real vector fields;
2. g is $C(M)$*-bilinear*: $g(fX, Y) = g(X, fY) = f\,g(X, Y)$, if $f \in C(M)$;
3. $g(X, X) \geq$ for X real, with $g(X, X) = 0 \implies X = 0$ in $\mathfrak{X}(M)$.

The second condition entails that g is given by a continuous family of symmetric bilinear maps $g_x \colon T_x^{\mathbb{C}}M \times T_x^{\mathbb{C}}M \to \mathbb{C}$ or $g_x \colon T_xM \times T_xM \to \mathbb{R}$; the latter version is positive definite.

2.4 The existence of $\mathrm{Spin}^{\mathrm{c}}$ structures

Suppose that $n = 2m + 1 = \dim M$ is odd. If n is odd then the fibres of B are semisimple but not simple: $\mathbb{C}\mathrm{l}(T^*_xM) \cong M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C})$ and we shall restrict to the even subalgebras, $\mathbb{C}\mathrm{l}^0(T^*_xM) \cong M_{2^m}(\mathbb{C})$, by demanding *that* $c(\gamma)$ *act as the identity in all cases*. Then we may adopt the convention that

$$c(\kappa) := c(\kappa\gamma) \quad \text{when } \kappa \text{ is odd.}$$

Notice here that $\kappa\gamma$ is even; and $c(\gamma) = c(\gamma^2) = +1$ is required for consistency of this rule. We take $A = C(M)$, but for B we now take

$$B := \begin{cases} \Gamma(M, \mathbb{C}\mathrm{l}(T^*M)), & \text{if } \dim M \text{ is even,} \\ \Gamma(M, \mathbb{C}\mathrm{l}^0(T^*M)), & \text{if } \dim M \text{ is odd.} \end{cases} \tag{7}$$

The fibres of these bundles are *central simple algebras* of finite dimension 2^{2m} in all cases.

We classify the algebras B as follows. Taking

$$\underline{B} := \begin{cases} \{ B_x = \mathbb{C}\mathrm{l}(T_x^*M) : x \in M \}, & \text{if } \dim M \text{ is even} \\ \{ B_x = \mathbb{C}\mathrm{l}^0(T_x^*M) : x \in M \}, & \text{if } \dim M \text{ is odd} \end{cases}$$

to be the collection of fibres, we can say that $\underline{B}$ is a "continuous field of simple matrix algebras", which moreover is *locally trivial*. There is an invariant

$$\delta(\underline{B}) \in \mathrm{H}^3(M; \mathbb{Z})$$

for such fields, found by Dixmier and Douady. The *Dixmier–Douady* class $\delta(\underline{B})$ is described in [18]. Now we would like to find a bundle $S \to M$ such that there are natural isomorphisms

$$\mathrm{End}_A\left(\Gamma\left(M, S\right)\right) \cong B;\ \mathrm{End}\left(\Gamma\left(M, S\right)\right)_B \cong A. \tag{8}$$

If $x \in M$, take $p_x \in B_x$ to be a *projector of rank one*, that is,

$$p_x = p_x^* = p_x^2 \quad \text{and} \quad \mathrm{tr}\, p_x = 1.$$

On the left ideal $S_x := B_x p_x$, we introduce a hermitian scalar product

$$\langle a_x p_x \mid b_x p_x \rangle := \mathrm{tr}(p_x a_x^* b_x p_x). \tag{9}$$

Notice that the recipe

$$|a_x p_x\rangle\langle b_x p_x| : c_x p_x \mapsto (a_x p_x)(b_x p_x)^*(c_x p_x) = (a_x p_x b_x^*)(c_x p_x)$$

identifies $\mathcal{L}(S_x)$ —or $\mathcal{K}(S_x)$ in the infinite-dimensional case— with B_x, since the two-sided ideal $\mathrm{span}\{ a_x p_x b_x^* : a_x, b_x \in B_x \}$ equals B_x by simplicity.

Following proposition gives necessary and sufficient conductions of the existence of the *globally* defined the Hilbert spaces $S_x \cong \mathbb{C}^{2^m}$ such that $\mathcal{L}(S_x) \cong B_x$, for any $x \in M$.

Proposition 2.8. *[18] Let (M, g) be a compact Riemannian manifold. With $A = C(M)$ and B the algebra of Clifford sections given by* (7)*, the Dixmier–Douady class $\delta(\underline{B})$ vanishes, i.e., $\delta(\underline{B}) = 0$, if and only if there is a finitely generated projective A-module $\mathcal{S}$, carrying a selfadjoint action of B by A-linear operators, such that* $\mathrm{End}_A(S) \cong B$.

Let denote $\mathcal{S} = \Gamma(M, S)$ and $\mathcal{S}^\sharp = \mathrm{Hom}_A(\mathcal{S}, A)$. From the (8) it follows that if $\mathcal{S}^\sharp = \mathrm{Hom}_A(\mathcal{S}, A)$ then $\mathcal{S} \otimes_A \mathcal{S}^\sharp \cong B$ and $\mathcal{S}^\sharp \otimes_B \mathcal{S} \cong A$. Since $\mathcal{S}^\sharp \cong \Gamma(M, S^*)$ where $S^* \to M$ is the dual vector bundle to $S \to M$, we can write this equivalence fibrewise: $\mathcal{S}_x \otimes_{\mathbb{C}} \mathcal{S}_x^* = \mathrm{End}_{\mathbb{C}}(\mathcal{S}_x) \cong B_x$ and then $\mathcal{S}_x^* \otimes_{B_x} \mathcal{S}_x \cong \mathbb{C}$, for $x \in M$. To proceed, we explain how B acts on $\mathcal{S}^\sharp = \mathrm{Hom}_A(\mathcal{S}, A)$. The spinor module $\mathcal{S}$ carries an A-valued Hermitian pairing (9) given by the local scalar products defined in the construction of $\mathcal{S}$, that may be written

$$(\psi \mid \phi) \ :\ x \mapsto \langle \psi_x | \phi_x \rangle, \quad \text{for } x \in M. \tag{10}$$

We can identify elements of $\mathcal{S}^\sharp$ with "bra-vectors" $\langle \psi |$ using this pairing, namely, we define $\langle \psi |$ to be the map $\phi \mapsto (\psi \mid \phi) \in A$. Since A is unital, there is a "Riesz theorem"

for A-modules showing that all elements of $\mathcal{S}^\sharp$ are of this form. Now the left B-action is defined by

$$b\,\langle\psi| := \langle\psi| \circ \chi(b^!).$$

Recall that $b \mapsto \chi(b!)$ is a linear antiautomorphism of B. We also require triviality of the second Stiefel–Whitney class $\kappa(B) = w_2(TM) = w_2(T^*M)$ of the tangent (or cotangent) bundle described in [18]. The condition $\kappa(B) = 0$ is hold if

$$\mathcal{S}^\sharp \cong \mathcal{S} \quad \text{as } B\text{-}A\text{-bimodules.}$$

We now reformulate this condition in terms of a certain antilinear operator C; later on, in the context of spectral triples, we shall rename it to J.

Proposition 2.9. *[18] There is a B-A-bimodule isomorphism $\mathcal{S}^\sharp \cong \mathcal{S}$ if and only if there is an antilinear endomorphism C of $\mathcal{S}$ such that*

(a) $C(\psi\, a) = C(\psi)\,\bar{a}$ *for* $\psi \in \mathcal{S}$, $a \in A$;

(b) $C(b\,\psi) = \chi(\bar{b})\, C(\psi)$ *for* $\psi \in \mathcal{S}$, $b \in B$;

(c) C *is* antiunitary *in the sense that* $(C\phi \mid C\psi) = (\psi \mid \phi) \in A$, *for* $\phi, \psi \in \mathcal{S}$;

(d) $C^2 = \pm 1$ *on* $\mathcal{S}$ *whenever* M *is connected.*

The antilinear operator $C\colon \mathcal{S} \to \mathcal{S}$, which becomes an *antiunitary* operator on a suitable Hilbert-space completion of $\mathcal{S}$, is called the *charge conjugation*. It exists if and only if $\kappa(B) = 0$. What, then, are Spin$^{\mathbf{c}}$ and Spin structures on M? We choose on M a metric (without losing generality), and also an *orientation* ε, which organizes the action of B, in that a change $\varepsilon \mapsto -\varepsilon$ induces $c(\gamma) \mapsto -c(\gamma)$, which either

(i) reverses the $\mathbb{Z}_2$-grading of $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, in the even case; or

(ii) changes the action on $\mathcal{S}$ of each $c(\alpha)$ to $-c(\alpha)$, for $\alpha \in \mathcal{A}^1(M)$, in the odd case —recall that $c(\alpha) := c(\alpha\gamma)$ in the odd case.

Definition 2.10. [18] Let (M, ε) be a compact boundaryless orientable manifold, together with a chosen orientation ε. Let $A = C(M)$ and let B be specified as before (in terms of a fixed but arbitrary Riemannian metric on M). If $\delta(B) = 0$ in $\mathrm{H}^3(M;\mathbb{Z})$, a Spin$^{\mathbf{c}}$ *structure* on (M, ε) is an isomorphism class $[\mathcal{S}]$ of equivalence B-A-bimodules. If $\delta(B) = 0$ and if $\kappa(B) = 0$ in $\mathrm{H}^2(M;\mathbb{Z}_2)$, a pair $(\mathcal{S}, C)$ give data for a spin structure, when $\mathcal{S}$ is an equivalence B-A-bimodule such that $\mathcal{S}^\sharp \cong \mathcal{S}$, and C is a charge conjugation operator on $\mathcal{S}$. A *Spin structure* on (M, ε) is an isomorphism class of such pairs. A vector bundle $S \to M$ such that $\mathcal{S} = \Gamma(M, S)$ is said to be the *spinor bundle*.

2.5 The Spin connection

We now leave the topological level and introduce differential structure. Thus we replace $A = C(M)$ by $\mathcal{A} = C^\infty(M)$, and continuous sections Γ_{cont} by smooth sections Γ_{smooth}. Thus $\mathcal{S} = \Gamma_{\text{smooth}}(M, S)$ will henceforth denote the $\mathcal{A}$-module of *smooth* spinors.

Our treatment of Morita equivalence of *unital* algebras passes without change to the smooth level. We can go back with the functor $-\otimes_{C^\infty(M)} C(M)$, if desired.

Definition 2.11. [18] A *connection* on a (finitely generated projective) $\mathcal{A}$-module $\mathcal{E} = \Gamma(M,E)$ is a $\mathbb{C}$-linear map $\nabla\colon \mathcal{E} \to \mathcal{A}^1(M)\otimes_{\mathcal{A}}\mathcal{E} = \Gamma(M, T^*M\otimes E) \equiv \mathcal{A}^1(M,E)$, satisfying the Leibniz rule

$$\nabla(fs) = df\otimes s + f\,\nabla s.$$

It extends to an *odd derivation* of degree +1 on $\mathcal{A}^\bullet(M)\otimes_{\mathcal{A}}\mathcal{E} = \Gamma(M,\Lambda^\bullet T^*M\otimes E) \equiv \mathcal{A}^\bullet(M,E)$ with grading inherited from that of $\mathcal{A}^\bullet(M)$, leaving $\mathcal{E}$ trivially graded, so that $\nabla(\omega\wedge\sigma) = d\omega\wedge\sigma + (-1)^{|\omega|}\omega\wedge\nabla\sigma$ for $\omega\in\mathcal{A}^\bullet(M)$, $\sigma\in\mathcal{A}^\bullet(M,E)$.

Definition 2.12. [18] If $\mathcal{E}$ an $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued Hermitian pairing, we say that a *connection* ∇ on $\mathcal{E}$ is *Hermitian* if

$$\begin{aligned}(\nabla s\,|\,t) + (s\,|\,\nabla t) &= d\,(s\,|\,t), \quad \text{or, in other words,}\\ (\nabla_X s\,|\,t) + (s\,|\,\nabla_X t) &= X\,(s\,|\,t), \quad \text{for any } \textit{real}\ \ X\in\mathfrak{X}(M).\end{aligned}$$

If ∇,∇' are connections on $\mathcal{E}$, then $\nabla'-\nabla$ is an $\mathcal{A}$-module map: $(\nabla'-\nabla)(fs) = f(\nabla'-\nabla)s$, so that *locally*, over $U\subset M$ for which $E\big|_U \to U$ is trivial, we can write

$$\nabla = d+\alpha, \quad \text{where } \ \alpha\in\mathcal{A}^1(U,\mathrm{End}E).$$

Fact 2.13. [18] On $\mathfrak{X}(M) = \Gamma(M,TM)$ there is, for each Riemannian metric g, a *unique torsion-free connection that is compatible with* g:

$$g(\nabla X,Y) + g(X,\nabla Y) = d(g(X,Y)) \quad \text{for } \ X,Y\in\mathfrak{X}(M), \quad \text{or}$$

$$g(\nabla_Z X,Y) + g(X,\nabla_Z Y) = Z(g(X,Y)) \quad \text{for } \ X,Y,Z\in\mathfrak{X}(M).$$

The explicit formula for this connection is

$$\begin{aligned}2g(\nabla_X Y,Z) = X(g(Y,Z)) &+ Y(g(X,Z)) - Z(g(X,Y))\\ &+ g(Y,[Z,X]) + g(Z,[X,Y]) - g(X,[Y,Z]).\end{aligned} \tag{11}$$

It is called *Levi-Civita connection* associated to g. (The proof of existence consists in showing that the right hand side of this expression is $\mathcal{A}$-linear in Y and Z, and obeys a Leibniz rule with respect to X, so it gives a connection; and uniqueness is obtained by checking that metric compatibility and torsion freedom make the right hand side automatic.)
The *dual connection* on $\mathcal{A}^1(M)$ will also be called the "Levi-Civita connection". At the risk of some confusion, we shall use the same symbol ∇ for both of these Levi-Civita connections.

Definition 2.14. [18] On a spinor module $\mathcal{S} = \Gamma(M,S)$, a *spin*$^{\mathbf{c}}$*-connection* is any *Hermitian* connection $\nabla^S\colon \mathcal{S}\to\mathcal{A}^1(M)\otimes_{\mathcal{A}}\mathcal{S}$ which is compatible with the action of B in the following way:

$$\begin{aligned}&\nabla^S(c(\alpha)\psi) = c(\nabla\alpha)\psi + c(\alpha)\nabla^S\psi \quad \text{for } \ \alpha\in\mathcal{A}^1(M),\ \psi\in\mathcal{S}; \quad \text{or}\\ &\nabla^S_X(c(\alpha)\psi) = c(\nabla_X\alpha)\psi + c(\alpha)\nabla^S_X\psi \quad \text{for } \ \alpha\in\mathcal{A}^1(M),\ \psi\in\mathcal{S},\ X\in\mathfrak{X}(M),\end{aligned} \tag{12}$$

where $\nabla\alpha$ and $\nabla_X\alpha$ refer to the *Levi-Civita connection* on $\mathcal{A}^1(M)$.
If $(\mathcal{S}, C)$ are data for a spin structure, we say ∇^S is a *spin connection* if, moreover, each $\nabla_X : \mathcal{S} \to \mathcal{S}$ commutes with C whenever X is real.

$$\nabla^S(c(\alpha)\psi) = c(\nabla\alpha)\psi + c(\alpha)\nabla^S\psi.$$

Proposition 2.15. *[18] If $(\mathcal{S}, C)$ are data for a spin structure on M, then there is a unique Hermitian spin connection $\nabla^S : \mathcal{S} \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S}$, such that*

$$\nabla^S(c(\alpha)\psi) = c(\nabla\alpha)\psi + c(\alpha)\nabla^S\psi, \quad \text{for } \alpha \in \mathcal{A}^1(M),\ \psi \in \mathcal{S},$$

and such that $\nabla^S_X C = C\nabla^S_X$ for $X \in \mathfrak{X}(M)$ real.

2.6 Dirac operators

Suppose we are given a compact oriented (boundaryless) Riemannian manifold (M, ε) and a spinor module with charge conjugation $(\mathcal{S}, C)$, together with a Riemannian metric g, so that the Clifford action $c : \mathcal{B} \to \mathrm{End}_{\mathcal{A}}(\mathcal{S})$ has been specified. We can also write it as $\hat{c} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{S}, \mathcal{S})$ by setting $\hat{c}(\kappa \otimes \psi) := c(\kappa)\,\psi$.

Definition 2.16. [18] Using the inclusion $\mathcal{A}^1(M) \hookrightarrow \mathcal{B}$ —where in the odd dimensional case this is given by $c(\alpha) := c(\alpha\gamma)$, as before— we can form the composition

$$D\!\!\!\!/ := -i\,\hat{c} \circ \nabla^S \tag{13}$$

where

$$\mathcal{S} \xrightarrow{\nabla^S} \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S} \xrightarrow{\hat{c}} \mathcal{S},$$

so that $D\!\!\!\!/ : \mathcal{S} \to \mathcal{S}$ is $\mathbb{C}$-linear. This is the *Dirac operator* associated to $(\mathcal{S}, C)$ and g.

The $(-i)$ is included in the definition to make $D\!\!\!\!/$ symmetric (instead of skewsymmetric) as an operator on a Hilbert space, because we have chosen g to be positive definite, that is, $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = +2\,\delta^{\alpha\beta}$. Historically, $D\!\!\!\!/$ was introduced as $-i\gamma^\mu\delta_\mu = \gamma^\mu p_\mu$ where the p_μ are components of a 4-momentum, but in the Minkowskian signature.
Using local (coordinate or orthonormal) bases for $\mathfrak{X}(M)$ and $\mathcal{A}^1(M)$, we get nicer formulas:

$$D\!\!\!\!/\,\psi = -i\,\hat{c}(\nabla^S\psi) = -i\,c(dx^j)\nabla^S_{\partial_j}\psi = -i\,\gamma^\alpha\nabla^S_{E_\alpha}\psi. \tag{14}$$

The essential algebraic property of $D\!\!\!\!/$ is the *commutation relation*:

$$[D\!\!\!\!/, a] = -i\,c(da), \quad \text{for all} \quad a \in \mathcal{A} = C^\infty(M). \tag{15}$$

Indeed,

$$\begin{aligned}
[D\!\!\!\!/, a]\,\psi &= -i\,\hat{c}(\nabla^S(a\psi)) + ia\,\hat{c}(\nabla^S\psi) \\
&= -i\,\hat{c}(\nabla^S(a\psi) - a\,\nabla^S\psi) \\
&= -i\,\hat{c}(da \otimes \psi) = -i\,c(da)\,\psi, \quad \text{for } \psi \in \mathcal{S}.
\end{aligned}$$

As an operator, we can make sense of $[\not{D}, a]$ by conferring on $\mathcal{S}$ the structure of a Hilbert space: if we write $\det g := \det[g_{ij}]$ for short, then

$$\nu_g := \sqrt{\det g}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \in \mathcal{A}^n(M)$$

is the Riemannian volume form (for the given orientation ε and metric g). In the notation, we assume that all local charts are consistent with the given orientation, which just means that $\det[g_{ij}] > 0$ in any local chart. The scalar product on $\mathcal{S}$ is then given by

$$(\phi, \psi) \stackrel{\text{def}}{=} \int_M (\phi \mid \psi)\, \nu_g \quad \text{for } \phi, \psi \in \mathcal{S}. \tag{16}$$

On completion in the norm $\|\psi\| \stackrel{\text{def}}{=} \sqrt{(\psi, \psi)}$, we get the Hilbert space $H := L^2(M, S)$ of L^2-spinors on M. In the even case, $B = \Gamma(M, \mathbb{C}l(T^*M))$ contains the operator $\Gamma = c(\gamma)$ which extends to a selfadjoint unitary operator on H. It is known that $\big(C^\infty(M), L^2(M, S), \not{D}, C, \Gamma\big)$ (resp. $\big(C^\infty(M), L^2(M, S), \not{D}, C\big)$) is a spectral triple in case of even (resp. odd) dimension [18].

2.7 Definition of spectral triples

Definition 2.17. [18] A (unital) **spectral triple** $(\mathscr{A}, H, D)$ consists of:

- an *algebra* $\mathscr{A}$ with an involution $a \mapsto a^*$, equipped with a faithful representation on:
- a *Hilbert space* H; and also
- a *selfadjoint operator* D on $\mathcal{H}$, with dense domain $\operatorname{Dom} D \subset H$, such that $a(\operatorname{Dom} D) \subseteq \operatorname{Dom} D$ for all $a \in \mathcal{A}$,

satisfying the following two conditions:

- the operator $[D, a]$, defined initially on $\operatorname{Dom} D$, extends to a *bounded operator* on H, for each $a \in \mathcal{A}$;
- D has *compact resolvent*: $(D - \lambda)^{-1}$ is compact, when $\lambda \notin \operatorname{sp}(D)$.

For now, and until further notice, all spectral triples will be defined over unital algebras. The compact-resolvent condition must be modified if $\mathcal{A}$ is nonunital: as well as enlarging $\mathcal{A}$ to a unital algebra, we require only that the products $a(D - \lambda)^{-1}$, for $a \in \mathcal{A}$ and $\lambda \notin \operatorname{sp}(D)$, be compact operators.

2.8 The Dixmier trace

The Dixmier trace is the noncommutative analogue of integral over a manifold.

2.18. [36] The algebra $\mathcal{K}$ of compact operators on a separable, infnite-dimensional Hilbert space contains the ideal $\mathcal{L}^1$ of traceclass operators, on which $\|T\|_1 = \mathrm{Tr}|T|$ is a norm not to be confused with the operator norm $\|T\|$. Let $\sigma_n(T)$ be such that

$$\sigma_n(T) = \sup\left\{\|TP_n\|_1 \mid P_n \text{ is a projector of rank } n\right\}$$

There is a formula [**?**], coming from real interpolation theory of Banach spaces:

$$\sigma_n(T) = \left\{\inf\{\|R\|_1 + n\|S\| \mid R, S \in \mathcal{K},\ R + S = T\right\}.$$

If $T \in \mathcal{K}$ is a compact operator then σ_n can be defined as

$$\sigma_n = \sum_{i=1}^{n} \lambda_i$$

where $\{\lambda_i\}_{i \in N}$ is a decreasing ordered set of the operator $(T^*T)^{1/2}$ eigenvalues, i.e. $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq ....$ We can think of $\sigma_n(T)$ as the trace of $|T|$ with a *cutoff* at the scale n. This scale does not have to be an integer; for any scale $\lambda > 0$, we can *define*

$$\sigma_\lambda(T) = \inf\left\{\|R\|_1 + \lambda\|S\| \mid R, S \in \mathcal{K},\ R + S = T\right\}.$$

If $0 < \lambda \leq 1$, then $\sigma_\lambda(T) = \lambda\|T\|$. If $\lambda = n + t$ with $0 \leq t < 1$, one checks that

$$\sigma_\lambda(T) = (1 - t)\sigma(T) + t\sigma_{n+1}(T), \tag{17}$$

so $\lambda \mapsto \sigma_\lambda(T)$ is a piecewise linear, increasing, concave function on $(0, 1)$.

Each σ_λ is a norm by (17), and so satisfies the triangle inequality. It is proved in [36] that for positive compact operators, there is a triangle inequality in the opposite direction:

$$\sigma_\lambda(A) + \sigma_\mu(B) \leq \sigma_{\lambda+\mu}(A + B); \text{ if } A, B > 0. \tag{18}$$

It suffices to check this for integral values $\lambda = m$, $\mu = n$. If P_m, P_n are projectors of respective ranks m, n, and if $P = P_m \vee P_n$ is the projector with range $P_mH + P_nH$, then

$$\|AP_m\|_1 + \|BP_n\|_1 = Tr(P_mAP_m) + Tr(P_nBP_n) \leq Tr(P(A + B)P) \leq \|(A + B)P\|_1,$$

and (18) follows by taking supremum over P_m, P_n. Thus we have a sandwich of norms:

$$\sigma_\lambda(A + B) \leq \sigma_\lambda(A) + \sigma_\lambda(B) \leq \sigma_{2\lambda}(A + B) \text{ if } A, B \geq 0. \tag{19}$$

2.19. [36] *The Dixmier ideal*. The *first-order infinitesimals* can now be defined precisely as the following normed ideal of compact operators:

$$\mathcal{L}^{1+} = \left\{T \in \mathcal{K} \mid \|T\|_{1+} = \sup_{\lambda > e} \frac{\sigma_\lambda(T)}{\log \lambda} < \infty\right\},$$

that obviously includes the traceclass operators $\mathcal{L}^1$. (On the other hand, if $p > 1$ we have $\mathcal{L}^{1+} \subset \mathcal{L}^p$, where the latter is the ideal of those T such that $\mathrm{Tr}|T|^p < 1$, for which $\sigma_\lambda(T) = O(\lambda^{1-1/p})$.) If $T \in \mathcal{L}^{1+}$, the function $\lambda \mapsto \sigma_\lambda(T)/\log\lambda$ is continuous and bounded on the interval $[e,\infty)$, i.e., it lies in the C^*-algebra $C_b[e,\infty)$. We can then form the following Cesàro mean of this function:

$$\tau_\lambda(T) = \frac{1}{\log\lambda}\int_e^\lambda \frac{\sigma_u(T)}{\log u}\frac{du}{u}.$$

Then $\lambda \mapsto \tau_\lambda(T)$ lies in $C_b[e,\infty)$ also, with upper bound $\|T\|_{1+}$. From (19) we can derive that

$$\tau_\lambda(A) + \tau_\lambda(B) - \tau_\lambda(A+B) \le (\|A\|_{1+} + \|B\|_{1+}) \log 2 \frac{\log\log\lambda}{\log\lambda},$$

so that τ_λ is "asymptotically additive" on positive elements of $\mathcal{L}^{1+}$.

We get a true additive functional in two more steps. Firstly, let $\dot\tau(A)$ be the class of $\lambda \mapsto \tau_\lambda(A)$ in the quotient C^*-algebra $\mathcal{B} = C_b[e,\infty)/C_0[e,\infty)$. Then $\dot\tau$ is an additive, positive-homogeneous map from the positive cone of $\mathcal{L}^{1+}$ into $\mathcal{B}$, and $\dot\tau(UAU^{-1}) = \dot\tau(A)$ for any unitary U; therefore it extends to a linear map $\dot\tau : \mathcal{L}^{1+} \to \mathcal{B}$ such that $\dot\tau(ST) = \dot\tau(TS)$ for $T \in \mathcal{L}^{1+}$ and any S.

Secondly, we follow $\dot\tau$ with any state (i.e., normalized positive linear form) $\omega : \mathcal{B} \to \mathbb{C}$. The composition is a *Dixmier trace*:

$$\mathrm{Tr}_\omega(T) = \omega(\dot\tau(T)).$$

2.20. *The noncommutative integral.* Unfortunately, the C^*-algebra $\mathcal{B}$ is not separable and there is no way to *exhibit* any particular state. This problem can be finessed by noticing that a function $f \in C_b[e,\infty)$ has a limit $\lim_{\lambda\to\infty} f(\lambda) = c$ if and only if $\omega(f) = c$ does not depend on ω. Let us say that an operator $T \in \mathcal{L}^{1+}$ is *measurable* if the function $\lambda \mapsto \tau_\lambda(T)$ converges as $\lambda \to \infty$, in which case any $\mathrm{Tr}_\omega(T)$ equals its limit. We denote by $⨍ T$ the common value of the Dixmier traces:

$$⨍ T = \lim_{\lambda\to\infty} \tau_\lambda(T) \text{ if this limit exists.}$$

We call this value the *noncommutative integral* of T.

Note that if $T \in \mathcal{K}$ and $\sigma_n(T)/\log n$ converges as $n \to \infty$, then T lies in $\mathcal{L}^{1+}$ and is measurable.

Example 2.21. *Commutative case.* Let M be a compact spin-manifold, and let g be the Riemannian metric [36]. There is the Riemannian volume form Ω given by

$$\Omega = \sqrt{\det g(x)}dx^1 \wedge ... \wedge dx^n.$$

It is proven in [36] that for any $a \in C(M)$ following equation hold

$$\int_M a\Omega = \begin{cases} m!(2\pi)^m \fint a D\!\!\!/^{-2m}, & \text{if } \dim M = 2m \text{ is even,} \\ (2m+1)!!\pi^{m+1} \fint a|D\!\!\!/|^{-2m-1} & \text{if } \dim M = 2m+1 \text{ is odd.} \end{cases}$$

2.9 Regularity of spectral triples

The arguments of the previous section are not applicable to determine whether $[|D|, a]$ is bounded, in the case $r = 1$. This must be formulated as an assumption. In fact, we shall ask for much more: we want each element $a \in \mathcal{A}$, and each bounded operator $[D, a]$ too, to lie in the *smooth domain* of the following derivation.

Notation. We denote by δ the derivation on $B(\mathcal{H})$ given by taking the commutator with $|D|$. It is an unbounded derivation, whose domain is

$$\operatorname{Dom}\delta := \{\, T \in B(\mathcal{H}) : T(\operatorname{Dom}|D|) \subseteq \operatorname{Dom}|D|,\ [|D|, T] \text{ is bounded } \}.$$

We write $\delta(T) := [|D|, T]$ for $T \in \operatorname{Dom}\delta$.

Definition 2.22. [18] A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called *regular*, if for each $a \in \mathcal{A}$, the operators a and $[D, a]$ lie in $\bigcap_{k\in\mathbb{N}} \operatorname{Dom}\delta^k$.

Corollary 2.23. *[18] The standard commutative example $(C^\infty(M), L^2(M, S), D\!\!\!/\,)$ is a regular spectral triple.*

2.10 Pre-C*-algebras

If any spectral triple $(\mathcal{A}, \mathcal{H}, D)$, the algebra $\mathcal{A}$ is a (unital) $*$-algebra of bounded operators acting on a Hilbert space $\mathcal{H}$ [or, if one wishes to regard $\mathcal{A}$ abstractly, a faithful representation $\pi\colon \mathcal{A} \to B(\mathcal{H})$ is given]. Let A be the norm closure of $\mathcal{A}$ [or of $\pi(\mathcal{A})$] in $B(\mathcal{H})$: it is a C*-algebra in which $\mathcal{A}$ is a dense $*$-subalgebra.
A priori, the only functional calculus available for $\mathcal{A}$ is the holomorphic one:

$$f(a) := \frac{1}{2\pi i} \oint_\Gamma f(\lambda)(\lambda 1 - a)^{-1}\, d\lambda, \tag{20}$$

where Γ is a contour in $\mathbb{C}$ winding (once positively) around $\operatorname{sp}(a)$, and $\operatorname{sp}(a)$ means the spectrum of a in the C*-algebra A. To ensure that $a \in \mathcal{A}$ implies $f(a) \in \mathcal{A}$, we need the following property:
If $a \in \mathcal{A}$ has an inverse $a^{-1} \in A$, then in fact a^{-1} lies in $\mathcal{A}$ (briefly: $\mathcal{A} \cap A^\times = \mathcal{A}^\times$, where $\mathcal{A}^\times$ is the group of invertible elements of A). If this condition holds, then $\frac{1}{2\pi i}\oint_\Gamma f(\lambda)(\lambda 1 - a)^{-1}\, d\lambda$ is a limit of Riemannian sums lying in $\mathcal{A}$. To ensure convergence in $\mathcal{A}$ (they do converge in A), we need only ask that $\mathcal{A}$ be complete in some topology that is finer than the C^*-norm topology.

Definition 2.24. [18] A *pre-**C***-algebra* is a subalgebra of $\mathcal{A}$ of a C*-algebra A, which is stable under the holomorphic functional calculus of A.

If $\mathcal{A}$ is a nonunital algebra, we can always adjoin a unit in the usual way, and work with $\widetilde{\mathcal{A}} := \mathbb{C} \oplus \mathcal{A}$ whose unit is $(1, 0)$, and with its C^*-completion $\widetilde{A} := \mathbb{C} \oplus A$. Since the multiplication rule in $\widetilde{\mathcal{A}}$ is $(\lambda, a)(\mu, b) := (\lambda\mu, \mu a + \lambda b + ab)$, we see that $1 + a := (1, a)$ is invertible in $\widetilde{\mathcal{A}}$, with inverse $(1, b)$, if and only if $a + b + ab = 0$.

Lemma 2.25. *[18] If $\mathcal{A}$ is a unital, Fréchet pre-C^*-algebra, then so also is $M_n(\mathcal{A}) = M_n(\mathbb{C}) \otimes \mathcal{A}$.*

Lemma 2.26. *[18] The Schwartz algebra $\mathcal{S}(\mathbb{R}^n)$ is a nonunital pre-C^*-algebra.*

We state, without proof, two important facts about Fréchet pre-C^*-algebras.

Fact 2.27. [18] If $\mathcal{A}$ is a Fréchet pre-C^*-algebra and A is its C^*-completion, then $\mathrm{K}_j(\mathcal{A}) = \mathrm{K}_j(A)$ for $j = 0, 1$. More precisely, if $i \colon \mathcal{A} \to A$ is the (continuous, dense) inclusion, then $i_* \colon \mathrm{K}_j(\mathcal{A}) \to \mathrm{K}_j(A)$ is an surjective isomorphism, for $j = 0$ or 1.

This invariance of K-theory was proved by Bost [**?**]. For K_0, the spectral invariance plays the main role. For K_1, one must first formulate a topological K_1-theory is a category of "good" locally convex algebras (thus whose invertible elements form an open subset and for which inversion is continuous), and it is known that Fréchet pre-C^*-algebras are "good" in this sense.

Fact 2.28. [18] If $(\mathcal{A}, \mathcal{H}, D)$ is a regular spectral triple, we can confer on $\mathcal{A}$ the topology given by the seminorms

$$q_k(a) := \|\delta^k(a)\|, \quad q'_k(a) := \|\delta^k([D, a])\|, \quad \text{for each } k \in \mathbb{N}. \tag{21}$$

The completion $\mathcal{A}_\delta$ of $\mathcal{A}$ is then a Fréchet pre-C^*-algebra, and $(\mathcal{A}_\delta, \mathcal{H}, D)$ is again a regular spectral triple.

These properties of the completed spectral triple are due to Rennie [**?**]. We now discuss another result of Rennie, namely that such completed algebras of regular spectral triples are endowed with a C^∞ *functional calculus*.

Proposition 2.29. *[18] If $(\mathcal{A}, \mathcal{H}, D)$ is a regular spectral triple, for which $\mathcal{A}$ is complete in the Fréchet topology determined by the seminorms* (21)*, then $\mathcal{A}$ admits a C^∞-functional calculus. Namely, if $a = a^* \in \mathcal{A}$, and if $f \colon \mathbb{R} \to \mathbb{C}$ is a compactly supported smooth function whose support includes a neighbourhood of $\mathrm{sp}(a)$, then the following element $f(a)$ lies in $\mathcal{A}$:*

$$f(a) := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(s) \exp(isa)\, ds. \tag{22}$$

Before showing how this smooth functional calculus can yield useful results, we pause for a couple of technical lemmas on approximation of idempotents and projectors, in Fréchet pre-C^*-algebras. The first is an adaptation of a proposition of [**?**]

Lemma 2.30. *[18] Let $\mathcal{A}$ be an unital Fréchet pre-C^*-algebra, with C^*-norm $\|\cdot\|$. Then for each ε with $0 < \varepsilon < \frac{1}{8}$, we can find $\delta \leq \varepsilon$ such that, for each $v \in \mathcal{A}$ with $\|v - v^2\| < \delta$ and $\|1 - 2v\| < 1 + \delta$, there is an idempotent $e = e^2 \in \mathcal{A}$ such that $\|e - v\| < \varepsilon$.*

Lemma 2.30 says that in a unital Fréchet pre-C^*-algebra $\mathcal{A}$, an "almost idempotent" $v \in \mathcal{A}$ that is not far from being a projector (since $\|1 - 2v\|$ is close to 1) can be retracted to a genuine idempotent in $\mathcal{A}$. The next Lemma says that projectors in the C^*-completion of $\mathcal{A}$ can be approximated by projectors lying in $\mathcal{A}$.

Lemma 2.31. *[18] Let $\mathcal{A}$ be an unital Fréchet pre-C^*-algebra, whose C^*-completion is A. If $\tilde{q} = \tilde{q}^2 = \tilde{q}^*$ is a projector in A, then for any $\varepsilon > 0$, we can find a projector $q = q^2 = q^* \in \mathcal{A}$ such that $\|q - \tilde{q}\| < \varepsilon$.*

2.11 Real spectral triples

Recall that a spin structure on an oriented compact manifold (M, ε) is represented by a pair $(\mathcal{S}, C)$, where $\mathcal{S}$ is a $\mathcal{B}$-$\mathcal{A}$-bimodule and $C\colon \mathcal{S} \to \mathcal{S}$ is an antilinear map such that $C(\psi a) = C(\psi)\,\bar{a}$ for $a \in \mathcal{A}$; $C(b\,\psi) = \chi(\bar{b})\,C(\psi)$ for $b \in \mathcal{B}$; and, by choosing a metric g on M, which determines a Hermitian pairing on $\mathcal{S}$, we can also require that $(C\phi|C\psi) = (\phi|\psi) \in \mathcal{A}$ for $\phi, \psi \in \mathcal{S}$. $\mathcal{S}$ may be completed to a Hilbert space $\mathcal{H} = L^2(M, S)$, with scalar product $\langle\phi|\psi\rangle = \int_M (\phi|\psi), \nu_g$. It is clear that C extends to a bounded antilinear operator on $\mathcal{H}$ such that $\langle C\phi \mid C\psi\rangle = \langle\psi \mid \phi\rangle$ by integration with respect to ν_g, so that (the extended version of) C is *antiunitary* on $\mathcal{H}$. There are two tables of signs

n mod 8	0	2	4	6
$C^2 = \pm 1$	$+$	$-$	$-$	$+$
$C\not{D} = \pm\not{D}\,C$	$+$	$+$	$+$	$+$
$C\Gamma = \pm\Gamma C$	$+$	$-$	$+$	$-$

n mod 8	1	3	5	7
$C^2 = \pm 1$	$+$	$-$	$-$	$+$
$C\not{D} = \pm\not{D}\,C$	$-$	$+$	$-$	$+$

where n is a dimension of spin manifold (See [18] for details).

Definition 2.32. [18] A *real spectral triple* is a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, together with an antiunitary operator $J\colon \mathcal{H} \to \mathcal{H}$ such that $J(\operatorname{Dom} D) \subset \operatorname{Dom} D$, and $[a, Jb^*J^{-1}] = 0$ for all $a, b \in \mathcal{A}$.

Definition 2.33. [18] A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is *even* if there is a selfadjoint unitary operator Γ on $\mathcal{H}$ such that $a\Gamma = \Gamma a$ for all $a \in \mathcal{A}$, $\Gamma(\operatorname{Dom} D) = \operatorname{Dom} D$, and $D\Gamma = -\Gamma D$. If no such $\mathbb{Z}_2$-grading operator Γ is given, we say that the spectral triple is *odd*.

We have seen that in the standard commutative example, the *even* case arises when the auxiliary algebra $\mathcal{B}$ contains a natural $\mathbb{Z}_2$-grading operator, and this happens exactly when *the manifold dimension is even*. Now, the manifold dimension is determined by the spectral growth of the Dirac operator, and this spectral version of dimension may be used for noncommutative spectral triples, too. To make this more precise, we must look more closely at spectral growth.

2.12 Geometric conditions on spectral triples

We begin by listing a set of requirements on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, whose algebra $\mathcal{A}$ is unital but not necessarily commutative, such that $(\mathcal{A}, \mathcal{H}, D)$ provides a "spin geometry" generalization of our "standard commutative example" $(C^\infty(M), L^2(M, S), \not{D})$. Again we shall assume, for convenience, that D is invertible.

Condition 1 (Spectral dimension)**.** There is an *integer* $n \in \{1, 2, \dots\}$, called the spectral dimension of $(\mathcal{A}, \mathcal{H}, D)$, such that $|D|^{-1} \in \mathcal{L}^{n+}(\mathcal{H})$, and $0 < \operatorname{Tr}_\omega(|D|^{-n}) < \infty$ for any Dixmier trace Tr_ω.
When n is *even*, the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is also even: that is, there exists a selfadjoint unitary operator $\Gamma \in B(\mathcal{H})$ such that $\Gamma(\operatorname{Dom} D) = \operatorname{Dom} D$, satisfying $a\Gamma = \Gamma a$ for all $a \in \mathcal{A}$, and $D\Gamma = -\Gamma D$.

Condition 2 (Regularity)**.** For each $a \in \mathcal{A}$, the bounded operators a and $[D, a]$ lie in the smooth domain $\bigcap_{k \geq 1} \operatorname{Dom} \delta^k$ of the derivation $\delta \colon T \mapsto [|D|, T]$.
Moreover, $\mathcal{A}$ is complete in the topology given by the seminorms $q_k \colon a \mapsto \|\delta^k(a)\|$ and $q'_k \colon a \mapsto \|\delta^k([D, a])\|$. This ensures that $\mathcal{A}$ is a Fréchet pre-C^*-algebra.

Condition 3 (Finiteness)**.** The subspace of smooth vectors $\mathcal{H}^\infty := \bigcap_{k \in \mathbb{N}} \operatorname{Dom} D^k$ is a *finitely generated projective* left $\mathcal{A}$-module.
This is equivalent to saying that, for some $N \in \mathbb{N}$, there is a projector $p = p^2 = p^*$ in $M_N(\mathcal{A})$ such that $\mathcal{H}^\infty \cong \mathcal{A}^N p$ as left $\mathcal{A}$-modules.

Condition 4 (Real structure)**.** There is an antiunitary operator $J : \mathcal{H} \to \mathcal{H}$ satisfying $J^2 = \pm 1$, $JDJ^{-1} = \pm D$, and $J\Gamma = \pm \Gamma J$ in the even case, where the signs depend only on $n \bmod 8$ (and thus are given by the table of signs for the standard commutative examples).

$n \bmod 8$	0	2	4	6
$J^2 = \pm 1$	$+$	$-$	$-$	$+$
$JD = \pm DJ$	$+$	$+$	$+$	$+$
$J\Gamma = \pm \Gamma J$	$+$	$-$	$+$	$-$

$n \bmod 8$	1	3	5	7
$J^2 = \pm 1$	$+$	$-$	$-$	$+$
$JD = \pm DJ$	$-$	$+$	$-$	$+$

Moreover, $b \mapsto Jb^*J^{-1}$ is an antirepresentation of $\mathcal{A}$ on $\mathcal{H}$ (that is, a representation of the opposite algebra $\mathcal{A}^{\mathrm{op}}$), which commutes with the given representation of $\mathcal{A}$:

$$[a, Jb^*J^{-1}] = 0, \quad \text{for all } a, b \in \mathcal{A}.$$

Condition 5 (First order)**.** For each $a, b \in \mathcal{A}$, the following relation holds:

$$[[D, a], Jb^*J^{-1}] = 0, \quad \text{for all } a, b \in \mathcal{A}. \tag{23}$$

This generalizes, to the noncommutative context, the condition that D be a first-order differential operator.

Since

$$[[D,a],Jb^*J^{-1}] = [[D,Jb^*J^{-1}],a] + [D,\underbrace{[a,Jb^*J^{-1}]}_{=0}],$$

this is equivalent to the condition that $[a,[D,Jb^*J^{-1}]] = 0$.

Condition 6 (Orientation)**.** There is a Hochschild n-cycle

$$\mathbf{c} = \sum_j (a_j^0 \otimes b_j^0) \otimes a_j^1 \otimes \cdots \otimes a_j^n \in Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}),$$

such that

$$\pi_D(\mathbf{c}) \equiv \sum_j a_j^0 (Jb_j^{0*}J^{-1})\,[D,a_j^1]\dots[D,a_j^n] = \begin{cases} \Gamma, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases} \tag{24}$$

In many examples, including the noncommutative examples we shall meet in the next two sections, one can often take $b_j^0 = 1$, so that $\mathbf{c}$ may be replaced, for convenience, by the cycle $\sum_j a_j^0 \otimes a_j^1 \otimes \cdots \otimes a_j^n \in Z_n(\mathcal{A},\mathcal{A})$. In the commutative case, where $\mathcal{A}^{\mathrm{op}} = \mathcal{A}$, this identification may be justified: the product map $m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a homomorphism.
The data set $(\mathcal{A}, \mathcal{H}, D; \Gamma \text{ or } 1, J, \mathbf{c})$ satisfying these six conditions constitute a "noncommutative spin geometry". In the fundamental paper where these conditions were first laid out [11], Connes added one more nondegeneracy condition (Poincaré duality in K-theory) as a requirement. We shall not go into this matter here.

3 Topological constructions

This section is concerned with a topological construction of an infinitely listed covering projection from finitely listed ones.

3.1. Let $\mathcal{X}$ be a second-countable [27] locally compact connected Hausdorff space, and let

$$\mathcal{X} = \mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ... \tag{25}$$

be a sequence of finitely listed covering projections. Let $\{G_n = G\left(\mathcal{X}_n|\mathcal{X}\right)\}_{n\in\mathbb{N}}$ be groups of covering transformations. Let $\widehat{\mathcal{X}} = \varprojlim \mathcal{X}_n$, $\widehat{G} = \varprojlim G_n$ be inverse limits. The group $\widehat{G}$ has a topology defined by subgroups of finite index [25]. Let $\overline{G}$ be a discrete group algebraically isomorphic to $\widehat{G}$. For any $x \in \widehat{\mathcal{X}}$ let us define a map $\phi_x : \overline{G} \to \widehat{\mathcal{X}}$ be given by $g \mapsto gx$. We define a topological space $\overline{\mathcal{X}}$ as the set $\widehat{\mathcal{X}}$ with the final topology [5] such that the identical map $\mathrm{Id} : \widehat{\mathcal{X}} \to \overline{\mathcal{X}}$, and maps φ_x for any $x \in \widehat{\mathcal{X}}$ are continuous. Action of $\overline{G}$ on $\overline{\mathcal{X}}$ is free and properly discontinuous, so there is a natural regular covering projection $\overline{\pi} : \overline{\mathcal{X}} \to \mathcal{X}$. Let $\widetilde{\mathcal{X}}$ be a connected component of $\overline{\mathcal{X}}$, and let $G \subset \overline{G}$ be a maximal subgroup such that

$$G\widetilde{\mathcal{X}} = \widetilde{\mathcal{X}}. \tag{26}$$

For any $g \in \overline{G}$ a subgroup gGg^{-1} satisfies to (26), i.e. $gGg^{-1} = G$, whence G is a normal subgroup. Any connected component of a covering space is also a covering space, therefore $\widetilde{\pi} = \overline{\pi}|_{\widetilde{\mathcal{X}}} : \widetilde{\mathcal{X}} \to \mathcal{X}$ is a covering projection and $\mathcal{X} \approx \widetilde{\mathcal{X}}/G$, i.e. $\widetilde{\mathcal{X}} \to \mathcal{X}$ is a regular covering projection.

Definition 3.2. The space $\overline{\mathcal{X}}$ is said to be a *disconnected covering space* of the sequence (25), and the space $\widetilde{\mathcal{X}}$ is said to be a *connected covering space* of the sequence (25).

Definition 3.3. [10] Let $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a (topological) covering projection. A *fundamental domain* of G is an open connected set $\mathcal{D} \subset \widetilde{\mathcal{X}}$ such that

(a)

$$p\left(\mathfrak{cl}\left(\mathcal{D}\right)\right) = \mathcal{X} \tag{27}$$

(b)

$$g\mathcal{D} \bigcap \mathcal{D} = \emptyset \text{ for any nontrivial } g \in G. \tag{28}$$

Remark 3.4. The condition (a) of the Definition 3.3 is equivalent to

$$\widetilde{\mathcal{X}} = \bigcup_{g \in G} g \ \mathfrak{cl}\left(\mathcal{D}\right).$$

Remark 3.5. [10] When considering a Riemannian covering $\widetilde{M} \to M$ one may construct a fundamental by following way. For any point $x \in M$ there is the *cut loci*, which is an open set $\Omega_x \subset M$ such that $M \backslash \Omega_x$ is a set of measure 0 [10]. It means that $1_{\Omega_x} \in L^\infty\left(M\right)$ and

$$1_{\Omega_x} = 1_M. \tag{29}$$

From [10] it follows that for any $\widetilde{x} \in p^{-1}\left(x\right)$ there is the natural connected open subset $\widetilde{\Omega}_{\widetilde{x}}$ which is mapped homeomorphically on Ω_x and

$$\sum_{g \in G\left(\widetilde{M}|M\right)} g \ 1_{\widetilde{\Omega}_{\widetilde{x}}} = 1_{\widetilde{M}}. \tag{30}$$

The definition of the cut loci is contained in [10].

Definition 3.6. Let $\widetilde{\mathcal{X}} \to \mathcal{X}$ be a topological covering projection. A finite or countable family $\left\{\mathcal{U}_\iota \subset \mathcal{X}\right\}_{\iota \in I}$ of connected relatively compact open sets, such that

1.

$$\mathcal{U}_i \text{ is evenly covered by } \pi^{-1}\left(\mathcal{U}_\iota\right), \tag{31}$$

2.

$$\bigcup_{\iota \in I} \mathcal{U}_\iota = \mathcal{X} \text{ is a locally finite covering,} \tag{32}$$

3.

$$\bigcup_{\iota \in I \backslash \{\iota_0\}} \mathcal{U}_\iota \neq \mathcal{X}; \ \forall \iota_0 \in I. \tag{33}$$

is said to be a *fundamental covering* of $\widetilde{\mathcal{X}} \to \mathcal{X}$. Let us select a single connected subset $\widetilde{\mathcal{U}}_\iota \subset \widetilde{\mathcal{X}}$ which is mapped homeomorphically onto $\mathcal{U}_\iota$, and we require that the union $\bigcup_{\iota \in I} \widetilde{\mathcal{U}}_\iota$ is a connected set. The family $\left\{\widetilde{\mathcal{U}}_\iota \subset \widetilde{\mathcal{X}}\right\}_{\iota \in I}$ is said to be a *basis of the fundamental covering*.

Definition 3.7. Let $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a covering projection and let $\widetilde{\mathcal{U}} \subset \widetilde{\mathcal{X}}$ be a connected open set which is mapped homeomorphicaly onto $\mathcal{U} = \widetilde{\pi}\left(\widetilde{\mathcal{U}}\right)$. If $\varphi \in C_0(\mathcal{X})$ is such that $\varphi(\mathcal{X} \backslash \mathcal{U}) = \{0\}$ then a function $\widetilde{\varphi} \in C_0\left(\widetilde{\mathcal{X}}\right)$ given by

$$\widetilde{\varphi}(\widetilde{x}) = \begin{cases} \varphi(\widetilde{\pi}(\widetilde{x})) & \widetilde{x} \in \widetilde{\mathcal{U}} \\ 0 & \widetilde{x} \notin \widetilde{\mathcal{U}} \end{cases}$$

is said to be the $\widetilde{\mathcal{U}}$*-lift* of φ. Otherwise if $\widetilde{\varphi} \in C_0\left(\widetilde{\mathcal{X}}\right)$ is such that $\widetilde{\varphi}\left(\widetilde{\mathcal{X}} \backslash \widetilde{\mathcal{U}}\right) = \{0\}$ then a function $\varphi \in C_0(\mathcal{X})$ given by

$$\varphi(x) = \begin{cases} \widetilde{\varphi}(\widetilde{x}) & x \in \mathcal{U},\ \widetilde{x} \in \widetilde{\mathcal{U}},\ \widetilde{\pi}(\widetilde{x}) = x \\ 0 & x \notin \mathcal{U} \end{cases}$$

is said to be the *descent* of $\widetilde{\varphi}$ on $\mathcal{X}$.

Definition 3.8. Let $\left\{\widetilde{\mathcal{U}}_\iota \subset \widetilde{\mathcal{X}}\right\}_{\iota \in I}$ be a basis of the fundamental covering of $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X}$. Let $\sum_{\iota \in I} a_\iota = 1_{C_b(\mathcal{X})}$ be a partition of unity dominated by $\left\{\widetilde{\pi}\left(\widetilde{\mathcal{U}}_\iota\right)\right\}_{\iota \in I}$, and let $\widetilde{a}_\iota \in C_0\left(\widetilde{\mathcal{X}}\right)$ be the $\widetilde{\mathcal{U}}_\iota$-lift of a_ι for any $\iota \in I$. A partition of unity given by

$$1_{C_b(\widetilde{\mathcal{X}})} = \sum_{g \in G} \sum_{\iota \in I} g\widetilde{a}_\iota = \sum_{(g,\iota) \in G \times I} \widetilde{a}_{(g,\iota)} \tag{34}$$

where $G = G\left(\widetilde{\mathcal{X}} \mid \mathcal{X}\right)$ and $\widetilde{a}_{(g,\iota)} = ga_\iota$ is said to be *dominated* by $\sum_{\iota \in I} a_\iota = 1_{C_b(\mathcal{X})}$. We also say that (34) is the partition of unity is *dominated* by $\left\{\widetilde{\mathcal{U}}_\iota\right\}_{\iota \in I}$.

3.9. Let $\phi_{[1,0]} : [0,1] \to [0,1]$ be a continuous function such that

$$\phi_{[1,0]}(x) = \begin{cases} 1 & x \le \frac{1}{2} \\ 0 & x = 1. \end{cases}$$

Let $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a covering projection such that $\widetilde{\mathcal{X}}$ is a locally compact metric space. For any $\widetilde{x} \in \widetilde{\mathcal{X}}$ there is $r > 0$ such that the set $\widetilde{\mathcal{U}} = \left\{\widetilde{y} \in \widetilde{\mathcal{X}} \mid \mathfrak{dist}(\widetilde{y}, \widetilde{x}) \le r\right\}$ is mapped homeomorphicaly on $\widetilde{\pi}\left(\widetilde{\mathcal{U}}\right)$. There is a function $\phi^{\widetilde{x}}_{[1,0]} \in C_0\left(\widetilde{\mathcal{X}}\right)$ given by

$$\phi^{\widetilde{x}}_{[1,0]}(\widetilde{y}) = \begin{cases} \phi_{[1,0]}\left(\frac{\mathfrak{dist}(\widetilde{y},\widetilde{x})}{r}\right) & \widetilde{y} \in \widetilde{\mathcal{U}} \\ 0 & \widetilde{y} \notin \widetilde{\mathcal{U}}. \end{cases} \tag{35}$$

There is an open neighborhood $\widetilde{\mathcal{V}} = \left\{\widetilde{y} \in \widetilde{\mathcal{X}} \mid \mathfrak{dist}(\widetilde{y}, \widetilde{x}) < \frac{r}{2}\right\}$ of $\widetilde{x}$ such that $\phi^{\widetilde{x}}_{[1,0]}\left(\widetilde{\mathcal{V}}\right) = \{1\}$.

Definition 3.10. A quadruple $\left(\widetilde{x}, \phi^{\widetilde{x}}_{[1,0]}, \widetilde{\mathcal{U}}, \widetilde{\mathcal{V}}\right)$ is said to be a *test function subordinated to* $\widetilde{\pi}$.

3.11. If M is a C^∞-manifold then we can define smooth test functions. Let us select a C^∞ function $\phi_{[1,0]} : [0,1] \to [0,1]$ such that following conditions hold Let $\phi_{[1,0]} : [0,1] \to [0,1]$ be a continuous function such that

$$\phi_{[1,0]}(x) = \begin{cases} 1 & x \le \frac{1}{2} \\ \\ 0 & x \ge \frac{3}{4}. \end{cases}$$

If $\widetilde{\pi} : \widetilde{M} \to M$ is a covering projection then any point $\widetilde{x} \in \widetilde{M}$ has an open neighborhood $\widetilde{x} \in \widetilde{U} \subset \widetilde{M}$ such that

1. There is an infinitely differentiable diffeomorphism $\psi : \widetilde{U} \to \mathbb{R}^n$.
2. The restriction $\widetilde{\pi}|_{\widetilde{U}} : \widetilde{U} \to \widetilde{\pi}\left(\widetilde{U}\right)$ is a homeomorphism.

Suppose that ψ is such that $\psi(\widetilde{x}) = 0 \in \mathbb{R}^n$, and let $\|\cdot\|_{\mathbb{R}^n}$ be a norm on $\mathbb{R}^n$ given by

$$\left\| \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix} \right\|_{\mathbb{R}^n} = \sqrt{x_1^2 + ... + x_n^2}.$$

Let us introduce a C^∞ function $\phi^{\widetilde{x}}_{[1,0]} : \widetilde{M} \to \mathbb{R}$ given by

$$\phi^{\widetilde{x}}_{[1,0]}(\widetilde{y}) = \begin{cases} \phi_{[1,0]}\left(\|\psi(\widetilde{y})\|_{\mathbb{R}^n}\right) & \widetilde{y} \in \widetilde{U} \\ 0 & \widetilde{y} \notin \widetilde{U}. \end{cases} \tag{36}$$

If $\widetilde{V} = \left\{\widetilde{y} \in \widetilde{U} \mid \|\psi(\widetilde{y})\|_{\mathbb{R}^n} < \frac{1}{2}\right\}$ then $\phi^{\widetilde{x}}_{[1,0]}\left(\widetilde{V}\right) = \{1\}$.

Definition 3.12. In the situation 3.11 a quadruple $\left(\widetilde{x}, \phi^{\widetilde{x}}_{[1,0]}, \widetilde{\mathcal{U}}, \widetilde{\mathcal{V}}\right)$ is said to be a C^∞ *test function subordinated* to $\widetilde{\pi}$.

Definition 3.13. Let $\mathcal{X}$, $\mathcal{Y}$ be Hausdorff spaces, and let $\{\mathcal{U}_\iota \subset \mathcal{X}\}_{\iota \in I}$ by a family of open sets such that $\mathcal{X} = \bigcup_{\iota \in I} \mathcal{U}_\iota$. A family of continuous maps $\varphi^{\mathcal{U}_\iota} : \mathcal{U}_\iota \to \mathcal{Y}$ is said to be *coherent* if $\varphi^{\mathcal{U}_\iota}|_{\mathcal{U}_\iota \bigcap \mathcal{U}_{\iota'}} = \varphi^{\mathcal{U}_{\iota'}}|_{\mathcal{U}_\iota \bigcap \mathcal{U}_{\iota'}}$ For any coherent sequence there is the unique continuous map $\varphi : \mathcal{X} \to \mathcal{Y}$ such that $\varphi|_{\mathcal{U}_\iota} = \varphi^{\mathcal{U}_\iota}; \forall \iota \in I$. The map φ is said to be the *gluing* of $\left\{\varphi^{\mathcal{U}_\iota}\right\}_{\iota \in I}$. Let denote by $\varphi = \mathfrak{Gluing}\left(\left\{\varphi^{\mathcal{U}_\iota}\right\}_{\iota \in I}\right)$ the gluing of $\left\{\varphi^{\mathcal{U}_\iota}\right\}_{\iota \in I}$.

4 Noncommutative covering projections

In this section I follow to the [17].

4.1 Finite case

Definition 4.1. [17] If A is a C^*-algebra then an action of a group G is said to be *involutive* if $ga^* = (ga)^*$ for any $a \in A$ and $g \in G$.

Definition 4.2. [17] If $B \subset A$ is an inclusion of C^*-algebras, G is a finite group with an involutive action on A such that $B = A^G$, then there is a B-valued sesquilinear product on A given by

$$\langle a, b \rangle_A = \frac{1}{|G|} \sum_{g \in G} g(a^* b)$$

The structure of Hilbert B-module A is said to be the *induced by* G-action.

Remark 4.3. The Definition 4.2 complies with the Theorem 1.17.

Definition 4.4. [17] Let $\pi : A \to \widetilde{A}$ be an injective *-homomorphism of C^*-algebras, and let G be a finite group such that following conditions hold:

1. There is an involutive continuous action of G on $\widetilde{A}$ such that $\widetilde{A}^G = A$;
2. $\widetilde{A} \subset \mathcal{K}\left(\widetilde{A}_A\right)$ where structure of Hilbert A-module $\widetilde{A}_A$ is induced by G-action;
3. There is a finite or countable set I and indexed by I subsets $\{a_\iota\}_{\iota \in I}$, $\{b_\iota\}_{\iota \in I} \subset \widetilde{A}$ such that

$$\sum_{\iota \in I} a_\iota (g b_\iota) = \begin{cases} 1_{M(\widetilde{A})} & g \in G \text{ is trivial} \\ 0 & g \in G \text{ is not trivial} \end{cases}. \tag{37}$$

   where the sum of the series means the strict convergence [3].

Then π is said to be a *finite noncommutative covering projection*, G is said to be the *covering transformation group*. Denote by $G(\widetilde{A}|A) = G$. The algebra $\widetilde{A}$ is said to be the *covering algebra*, and A is called the *base algebra* of the covering projection. A triple $\left(A, \widetilde{A}, G\right)$ is also said to be a *noncommutative finite covering projection*.

Remark 4.5. The article [17] contains the comprehensive foundation of the Definition 4.4.

Definition 4.6. Let $\left(A, \widetilde{A}, G\right)$ be a noncommutative finite covering projection. Algebra $\widetilde{A}$ is a finitely generated projective Hilbert A-modules with induced by G-action sesquilinear product given by

$$\langle a, b \rangle_{\widetilde{A}} = \frac{1}{|G|} \sum_{g \in G} g(a^* b) \tag{38}$$

We say that the structure of Hilbert A-module is *induced by the covering projection* $\left(A, \widetilde{A}, G\right)$. Henceforth we shall consider $\widetilde{A}$ as a right A-module.

4.7. Let $\left(A, \widetilde{A}, G\right)$ be a noncommutative finite covering projection. If $\widetilde{a} \in \widetilde{A}$ then $\widetilde{a} = a + p$ where $a = \frac{1}{|G|}\sum_{g \in G} g\widetilde{a}$ and $p = \widetilde{a} - a$. It is clear that $\sum_{g\in G} gp = 0$ and for any G-invariant $b \in \widetilde{A}$ we have

$$\langle b, p \rangle_{\widetilde{A}} = \frac{1}{|G|}\widetilde{b}\sum_{g \in G} gp = 0.$$

Otherwise the set of G-invariant elements is just a sublagebra $A \subset \widetilde{A}$. So $\widetilde{A}_A$ can be decomposed into the direct orthogonal sum, i.e.

$$\widetilde{A}_A = A \oplus P;\ A \perp P \text{ ,i.e. } \langle \widetilde{b}, p \rangle_{\widetilde{A}} = 0;\ \text{for any } a \in A;\ p \in P. \tag{39}$$

Example 4.8. *Finite covering projections of the circle* S^1. There is the universal covering projection $\widetilde{\pi} : \mathbb{R} \to S^1$. Let $\widetilde{\mathcal{U}}_1, \widetilde{\mathcal{U}}_2 \subset \mathbb{R}$ be such that

$$\widetilde{\mathcal{U}}_1 = (-\pi - 1/2, 1/2),\ \widetilde{\mathcal{U}}_2 = (-1/2, \pi + 1/2). \tag{40}$$

For any $i \in \{1,2\}$ the set $\mathcal{U}_i = \widetilde{\pi}(\widetilde{\mathcal{U}}_i) \subset S^1$ is open, connected and evenly covered. Since $S^1 = \mathcal{U}_1 \bigcup \mathcal{U}_2$ there is a partition of unity a_1, a_2 dominated by $\{\mathcal{U}_i\}_{i\in\{1,2\}}$ [27], i.e. $a_i : S^1 \to [0,1]$ are such that

$$\left\{\begin{array}{ll} a_i(x) > 0 & x \in \mathcal{U}_i \\ a_i(x) = 0 & x \notin \mathcal{U}_i \end{array}\right. ;\ i \in \{1,2\}.$$

and $a_1 + a_2 = 1_{C(S^1)}$. From the Proposition 1.38 is follows that we can select smooth partition of unity, i.e. $a_1, a_2 \in C^\infty(S^1)$. If $e_1, e_2 \in C^\infty(S^1)$ are given by

$$e_i = \sqrt{a_i};\ i = 1,2; \tag{41}$$

then

$$(e_1)^2 + (e_2)^2 = 1_{C_0(S^1)}.$$

If $\widetilde{e}_i \in C_0(\mathbb{R})$ are given by

$$\widetilde{e}_i(\widetilde{x}) = \left\{\begin{array}{ll} e_i(\widetilde{\pi}(\widetilde{x})) > 0 & \widetilde{x} \in \widetilde{\mathcal{U}}_i \\ 0 & \widetilde{x} \notin \widetilde{\mathcal{U}}_i \end{array}\right. ;\ i \in \{1,2\} \tag{42}$$

then there is a pointwise (=weak) convergence of the following series

$$\sum_{i=1,2;\ n \in \mathbb{Z}} n \cdot \widetilde{e}_i^2 = 1_{C_b(\mathbb{R}) = M(C_0(\mathbb{R}))},$$

where $n \cdot -$ means the natural action of $G(\mathbb{R}|S^1) \approx \mathbb{Z}$ on $C_0(\mathbb{R})$. Let $\overline{\pi}^n : \mathcal{X}_n \to S^1$ be an n-listed covering projection then $G(\mathcal{X}_n|S^1) \approx \mathbb{Z}_n$. It is well known that $\mathcal{X}_n \approx S^1$ but we use the $\mathcal{X}_n$ notion for clarity. There is a sequence of covering projections $\mathbb{R} \xrightarrow{\pi^n} \mathcal{X}_n \to S^1$.

If $\mathcal{U}_i^n = \pi^n\left(\widetilde{\mathcal{U}}_i\right)$ then $\mathcal{U}_i^n \bigcap g\mathcal{U}_i^n = \emptyset$ for any nontrivial $g \in G(\mathcal{X}_n, S^1)$. If $e_i^n \in C(\mathcal{X}_n)$ is given by

$$e_i^n(\pi^n(\widetilde{x})) = \begin{cases} \widetilde{e}_i(\widetilde{x}) & \pi^n(\widetilde{x}) \in \mathcal{U}_i^n \\ 0 & \pi^n(\widetilde{x}) \notin \mathcal{U}_i^n \end{cases} ;\ i \in \{1,2\} \tag{43}$$

then

$$\sum_{i \in \{1,2\};\ g \in G(\mathcal{X}_n, S^1)} g\left(e_i^n\right)^2 = 1_{C_0(\mathcal{X}_n)};$$

$$e_i^n(ge_i^n) = 0; \ \text{ for any notrivial } g \in G(\mathcal{X}_n|S^1).$$

If $I_n = G(\mathcal{X}_n|S^1) \times \{1,2\}$ and

$$e_\iota^n = ge_i^n;\ \text{where } \iota = (g,i) \in I_n \tag{44}$$

then

$$\sum_{\iota \in I_n} e_\iota^n(ge_\iota^n) = \begin{cases} 1_{C_0(\mathcal{X}_n)} & g \in G(\mathcal{X}_n|S^1) \text{ is trivial} \\ 0 & g \in G(\mathcal{X}_n|S^1) \text{ is not trivial} \end{cases}. \tag{45}$$

So a natural *-homomorphism $\pi : C(S^1) \to C(\mathcal{X}_n)$ satisfies the condition 3 of the Definition 4.4. Otherwise $C(\mathcal{X}_n) \approx C(S^1)^n$ as $C(S^1)$-module, i.e. $C(\mathcal{X}_n)$ is a finitely generated projective left and right $C(S^1)$-module. So a triple $\left(C_0(S^1), C_0(\mathcal{X}_n), \mathbb{Z}_n\right)$ is a finite noncommutative covering projection.

Example 4.9. *Finite covering projections of locally compact spaces.* The Example 4.8 can be generalized. Let $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a topological finitely listed covering projection such that $\mathcal{X}$ is a second-countable locally compact Hausdorff space. Suppose that both $\widetilde{\mathcal{X}}$ and $\mathcal{X}$ are connected spaces. There is an involutive continuous action of the covering transformation group $G = G\left(\widetilde{\mathcal{X}}|\mathcal{X}\right)$ on $C_0\left(\widetilde{\mathcal{X}}\right)$ arising from the action of G on $\widetilde{\mathcal{X}}$, and $C_0\left(\mathcal{X}\right) = C_0\left(\widetilde{\mathcal{X}}\right)^G$, i.e. condition 1 of the Definition 4.4 hold. Let $\left\{\widetilde{\mathcal{U}}_\iota \subset \widetilde{\mathcal{X}}\right\}_{\iota \in I}$ be a *basis of the fundamental covering*, and let $1_{C_b\left(\widetilde{\mathcal{X}}\right)} = \sum_{g \in G}\sum_{\iota \in I} g\widetilde{a}_\iota = \sum_{(g,\iota) \in G \times I} \widetilde{a}_{(g,\iota)}$ be a partition of unity dominated by $\left\{\widetilde{\mathcal{U}}_\iota\right\}_{\iota \in I}$. If $\widetilde{e}_{(g,\iota)} \in C_0\left(\widetilde{\mathcal{X}}\right)$ is given by

$$\widetilde{e}_{(g,\iota)} = \sqrt{g\widetilde{a}_\iota}. \tag{46}$$

then

$$\sum_{(g',\iota) \in G \times I} \widetilde{e}_{(g',\iota)}\left(g\widetilde{e}_{(g',\iota)}\right) = \begin{cases} 1_{M\left(C_0\left(\widetilde{\mathcal{X}}\right)\right)} & g \in G \text{ is trivial} \\ 0 & g \in G \text{ is not trivial} \end{cases}.$$

So condition 3 of the Definition 4.4 hold. If $\varphi \in C_0\left(\widetilde{\mathcal{X}}\right)$ is any function then

$$\varphi = \sum_{(g,\iota) \in G \times I} \varphi\widetilde{e}_{(g,\iota)}\widetilde{e}_{(g,\iota)} = \varphi\sqrt{\widetilde{e}_{(g,\iota)}}\sqrt{\widetilde{e}_{(g,\iota)}}\widetilde{e}_{(g,\iota)}$$

and from above equation and definition of $\widetilde{e}_{(g,\iota)}$ it follows that

$$\varphi = \sum_{\iota \in I} \left(\left\langle \varphi, \sqrt{\widetilde{e}_{(g,\iota)}} \right\rangle_{C_0(\widetilde{\mathcal{X}})} \sqrt{\widetilde{e}_{(g,\iota)}} \right) \rangle \langle \, \widetilde{e}_{(g,\iota)},$$

i.e. φ is an infinite sum of rank one operators. The sum is norm convergent, whence $\varphi \in \mathcal{K}\left(C_0 \left(\widetilde{\mathcal{X}} \right)_{C_0(\mathcal{X})} \right)$, and condition 2 of the Definition 4.4 hold. So all conditions of the definition 4.4 hold, whence the triple $\left(C_0\left(\mathcal{X}\right), C_0\left(\widetilde{\mathcal{X}}\right), G\left(\widetilde{\mathcal{X}}|\mathcal{X}\right) \right)$ is a finite noncommutative covering projection.

Example 4.10. *A finite covering projection of a noncommutative torus.* A noncommutative torus [36] A_θ is an universal unital C^*-algebra generated by two unitary elements ($u, v \in U(A_\theta)$) with one relation given by

$$uv = e^{2\pi i \theta} vu, \ (\theta \in \mathbb{R} \backslash \mathbb{Q}). \tag{47}$$

Let $\overline{\pi} : A_\theta \to A_{\theta'}$ be a *-homomorphism such that:

- There are $m, n, k \in \mathbb{N}$ such that $\theta' = \frac{\theta + 2\pi k}{mn}$;
- $A_{\theta'}$ is generated by $u_m, v_n \in U(A_{\theta'})$ and $\overline{\pi}$ is given by

$$u \mapsto u_m^m; \ v \mapsto v_n^n. \tag{48}$$

There is a continuous involutive action of $G = \mathbb{Z}_m \times \mathbb{Z}_n$ on $A_{\theta'}$ given by

$$\left(\overline{p}, \overline{q}\right) u_m = u_m e^{\frac{2\pi i p}{m}}, \ \left(\overline{p}, \overline{q}\right) v_n = v_n e^{\frac{2\pi i q}{n}}; \ \forall \left(\overline{p}, \overline{q}\right) \in G = \mathbb{Z}_m \times \mathbb{Z}_n,$$

and $A_\theta = A_{\theta'}^G$, i.e. the condition 1 of the Definition 4.4 hold. If $\{e_\iota^m\}_{\iota \in I_m}$ and $\{e_\iota^n\}_{\iota \in I_n}$ are given by (44) then from (45) it follows that

$$\begin{gathered} \sum_{\iota \in I_m} e_\iota^m(u_m) \left(\overline{k} e_\iota^m(u_m) \right) \left(\text{resp. } \sum_{\iota \in I_n} e_\iota^n(v_n) \left(\overline{k} e_\iota^n(v_n) \right) \right) = \\ = \left\{ \begin{array}{cc} 1_{A_{\theta'}} & \overline{k} = \overline{0} \\ 0 & \overline{k} \neq \overline{0} \end{array} \right. \text{ where } \overline{k} \in \mathbb{Z}_m \text{ (resp. } \overline{k} \in \mathbb{Z}_n\text{)}. \end{gathered} \tag{49}$$

If $I = I_m \times I_n$ and $\{e_\iota \in A_{\theta'}\}_{\iota \in I}$, $\{e'_\iota \in A_{\theta'}\}_{\iota \in I}$ are given by

$$e_\iota = e_{\iota_1}^m(u_m) e_{\iota_2}^n(v_n); \ e'_\iota = e_{\iota_2}^n(v_n) e_{\iota_1}^m(u_m); \ \iota_1 \in I_m, \ \iota_2 \in I_n, \ \iota = (\iota_1, \iota_2) \in I$$

then from (49) it follows that

$$\sum_{\iota \in I} e'_\iota e_\iota = \sum_{\iota_2 \in I_n} \left(e_{\iota_2}^n(v_n) \left(\sum_{\iota_1 \in I_m} e_k^m(u_m) e_k^m(u_m) \right) e_{\iota_2}^n(v_n) \right) = \sum_{\iota_2 \in I_n} \left(e_{\iota_2}^n(v_n) 1 e_{\iota_2}^n(v_n) \right) = 1.$$

If $g = (\overline{p}, \overline{q}) \in \mathbb{Z}_m \times \mathbb{Z}_n$ is such that $\overline{p} \neq \overline{0}$ then from (49) it follows that

$$\sum_{\iota \in I} e'_\iota(g e_\iota) = \sum_{\iota_2 \in I_n} \left(e^n_{\iota_2}(v_n) \left(\sum_{\iota_1 \in I_m} e^m_{\iota_1}(u_m) \left(\overline{p} e^m_{\iota_1}(u_n) \right) \right) \left(\overline{q} e^n_{\iota_2}(v_n) \right) \right) =$$

$$= \sum_{\iota_2 \in I_n} e^n_{\iota_2}(v_n) 0 \left(\overline{q} e^n_{\iota_2}(v_n) \right) = 0.$$

If $g = (\overline{p}, \overline{q}) \in \mathbb{Z}_m \times \mathbb{Z}_n$ is such that $\overline{p} = \overline{0}$ and $\overline{q} \neq \overline{0}$ then from (49) it follows that

$$\sum_{\iota \in I} e'_\iota(g e_\iota) = \sum_{\iota_2 \in I_n} \left(e^n_{\iota_2}(v_n) \left(\sum_{\iota_1 \in I_m} e^m_{\iota_1}(u_m) e^m_{\iota_1} k(u_m) \right) \left(\overline{q} e^n_{\iota_2}(v_n) \right) \right) =$$

$$= \sum_{\iota_2 \in I_n} e^n_{\iota_2}(v_n) 1 \left(\overline{q} e^n_{\iota_2}(v_n) \right) = \sum_{\iota_2 \in I_n} e^n_{\iota_2}(v_n) \left(\overline{q} e^n_{\iota_2}(v_n) \right) = 0.$$

From above equations it follows that

$$\sum_{\iota \in I,\ g \in \mathbb{Z}_m \times \mathbb{Z}_n} e'_\iota(g e_\iota) = \begin{cases} 1 & g \in \mathbb{Z}_m \times \mathbb{Z}_n \text{ is trivial} \\ 0 & g \in \mathbb{Z}_m \times \mathbb{Z}_n \text{ is not trivial} \end{cases}, \tag{50}$$

i.e. condition 3 of the Definition 4.4 hold. Otherwise $A_{\theta'} \approx A_\theta^{mn}$ as right and left A_θ module, i.e. condition 2 of the Definition 4.4 hold. So the triple $(A_\theta, A_{\theta'}, \mathbb{Z}_m \times \mathbb{Z}_n)$ is a noncommutative finite covering projection.

Example 4.11. *Boring example*. Let A (resp. G) be *any* unital C^*-algebra (resp. finite group.). Let $\widetilde{A} = \oplus_{g \in G} A_g$ where $A_g \approx A$ for any $g \in G$. Let $1_{A_g} \in G$ be the unity of A_g. Then $\widetilde{A}$ is a finitely generated left and right A-module. Action of G is given by $g_1 A_{g_2} = A_{g_1 g_2}$. We have

$$\sum_{g \in G} 1_{A_g} 1_{A_g} = 1_{\widetilde{A}},$$

$$\sum_{i = 1, \dots, n} 1_{A_g}(g 1_{A_g}) = 0 \ \forall g \in G \ (g \text{ is nontrivial}). \tag{51}$$

So a triple $\left(A, \oplus_{g \in G} A_g, G\right)$ is a finite noncommutative covering projection. This example is boring since it does not reflect properties of A and this projection can be constructed for any finite group.

Definition 4.12. A ring is said to be *irreducible* if it is not a direct sum of more than one nontrivial ring. A finite covering projection $(A, \widetilde{A}, G)$ is said to be *irreducible* if both A and $\widetilde{A}$ are irreducible. Otherwise $(A, \widetilde{A}, G)$ is said to be *reducible*.

Remark 4.13. Any reducible finite covering projection is boring. Covering projections from examples 4.8 - 4.10 are irreducible.

Definition 4.14. Let
$$A = A_0 \xrightarrow{\pi^1} A_1 \xrightarrow{\pi^2} ... \xrightarrow{\pi^n} A_n \xrightarrow{\pi^{n+1}} ...$$
be a finite or countable sequence of C^*-algebras and *-homomorphisms such that for any $i > 0$ there is a finite noncommutative covering projection (A_{i-1}, A_i, G_i). The sequence is said to be *composable* if following conditions hold:

1. Any composition $\pi_{i_1} \circ ... \circ \pi_{i_0+1} \circ \pi_{i_0} : A_{i_0} \to A_{i_1}$ corresponds to the noncommutative covering projection $\left(A_{i_0}, A_{i_1}, G\left(A_{i_1} | A_{i_0}\right)\right)$;
2. There is the natural exact sequence of covering transformation groups
$$\{e\} \to G\left(A_{i+2}|A_{i+1}\right) \xrightarrow{\iota} G\left(A_{i+2}|A_i\right) \xrightarrow{\pi} G\left(A_{i+1}|A_i\right) \to \{e\}$$
for any $i > 0$.

Example 4.15. Let
$$\mathcal{X} = \mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ...$$
be a finite or countable sequence of second-countable locally compact Hausdorff spaces and regular finitely listed topological covering projections. From the Example 4.9 it follows that for any i there is a finite noncommutative covering projection $\left(C\left(\mathcal{X}_{i-1}\right), C\left(\mathcal{X}_i\right), G_i\right)$. Since a composition of two regular finitely listed topological covering projections is also regular finitely listed topological covering projection the sequence
$$C\left(\mathcal{X}\right) = C\left(\mathcal{X}_0\right) \to C\left(\mathcal{X}_1\right) \to ... \to C\left(\mathcal{X}_n\right) \to ...$$
is composable.

4.2 Infinite case

This section is concerned with a noncommutative generalization of the described in the Section 3 construction. Let
$$A = A_0 \xrightarrow{\pi^1} A_1 \xrightarrow{\pi^2} ... \xrightarrow{\pi^n} A_n \xrightarrow{\pi^{n+1}} ... \tag{52}$$
be a sequence of *-homomorphisms which correspond to irreducible noncommutative finite covering projections, and suppose that (52) is composable. Let $G_n = G(A_n|A)$ be covering transformations groups, where $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ there is the natural group epimorphism $h_n : \overline{G} = \varprojlim G_m \to G_n$.

Definition 4.16. A sequence (52) is said to be *irreducible* if A_n is irreducible for any $n \in \mathbb{N}^0$.

4.17. Algebras $\{A_n\}_{n\in\mathbb{N}^0}$ are finitely generated projective Hilbert A-modules with sesquilinear product given by (38), i.e.
$$\langle a, b\rangle_{A_n} = \frac{1}{|G_n|} \sum_{g\in G_n} g(a^* b) \tag{53}$$

From (39) it follows that there A_n can be decomposed to a direct sum of orthogonal Hilbert A-modules

$$A_n = A_{n-1} \oplus P_n, \tag{54}$$

i.e. $\langle a, p \rangle_{A_n} = 0$ for any $a \in A_{n-1}$, $p \in P_n$.

Definition 4.18. A sequence $\{a_n \in A_n\}_{n \in \mathbb{N}^0}$ such that following conditions hold:

$$a_{n+1} = \frac{a_n}{|G(A_{n+1}|A_n)|} + p_{n+1}, \; p_{n+1} \in P_{n+1}; \tag{55}$$

$$\text{A sequence } \left\{ \langle a_n, a_n \rangle_{A_n} \in A \right\}_{n \in \mathbb{N}} \text{ is norm convergent as } n \to \infty \tag{56}$$

is said to be *coherent*.

Remark 4.19. The condition (55) is equivalent to

$$a_n = \sum_{g \in G(A_{n+1}|A_n)} g a_{n+1}. \tag{57}$$

Remark 4.20. Informally $\frac{a_n}{|G(A_{n+1}|A_n)|}$ means an infinitesimally small coefficient, because $\frac{a_n}{|G(A_{n+N}|A_n)|} \to 0$ as $N \to \infty$. If we denote $P_0 = A$ then $A_n = \bigoplus_{i \in \{0,\dots,n\}} P_i$, where P_i is finitely generated projective A module. Otherwise for any $n \in \mathbb{N}^0$ there is an inclusion $P_n \subset \mathcal{H}_A$ into the Hilbert space over A (Definition 4.2). From the Definition 4.18 it follows that

$$a_n = \sum_{k=0}^{n} \frac{p_k}{|G(A_n|A_k)|}$$

where $p_i \in \mathcal{H}_A$. For any $k \in \mathbb{N}^0$ there is an infinitesimally small element $p^k \in \mathcal{H}_A$ given by the sequence $\left\{ p_n^k \in A \right\}_{n \in \mathbb{N}}$ where

$$p_n^k = \left\{ \begin{array}{cc} 0 & n < k \\ \frac{p_k}{|G(A_n|A_k)|} & n \geq k \end{array} \right. . \tag{58}$$

Otherwise from (58) it follows that

$$\langle a_n, a_n \rangle_{A_n} = \sum_{k=0}^{\infty} \left(p_n^k \right)^* p_n^k,$$

$$\lim_{n \to \infty} \langle a_n, a_n \rangle_{A_n} = \lim_{n \to \infty} \sum_{k=0}^{\infty} \left(p_n^k \right)^* p_n^k$$

or

$$\lim_{n \to \infty} \langle a_n, a_n \rangle_{A_n} = \sum_{k=0}^{\infty} \left(p^k \right)^* p^k.$$

where p^k is infinitesimally small for any $k \in \mathbb{N}$. As well as in the Section 1.5 there is a representation by the sum of infinitesimally small elements.

4.21. There is an action of $\overline{G} = \varprojlim G_n$ on the linear space X' of coherent sequences arising from actions of G_n on A_n. There is the unique sesquilinear A-valued product $\langle\cdot,\cdot\rangle_{X'}$ on X' such that $\langle\{a_n\},\{b_n\}\rangle_{X'} = \lim_{n\to\infty}\langle a_n, b_n\rangle_{A_n}$. If $\mathcal{I} = \{x \in X' \mid \langle x, x\rangle_{X'} = 0\}$ and $X'' = X'/\mathcal{I}$ then there is a norm on X'' given by $\|x\| = \sqrt{\langle x, x\rangle_{X'}}$. Let $\overline{X}_A$ be the norm completion of X''. There is the unique sesquilinear A-valued product $\langle\cdot,\cdot\rangle_{\overline{X}_A}$ on $\overline{X}_A$ arising from $\langle\cdot,\cdot\rangle_{X'}$, so $\overline{X}_A$ is a Hilbert A-module. There is an action $\overline{G}$ on $\overline{X}_A$ arising from the action of $\overline{G}$ on X'.

Remark 4.22. As well as in the Remark 4.20 the inner product can be represented by the sum

$$\lim_{n\to\infty}\langle a_n, b_n\rangle_{A_n} = \sum_{k=0}^{\infty}\left(p^k\right)^* q^k.$$

where $p^k, q^k \in \mathcal{H}_A$ are infinitesimally small for any $k \in \mathbb{N}^0$.

Definition 4.23. Any coherent sequence $\{a_n \in A_n\}_{n\in\mathbb{N}^0}$ naturally gives the unique element $\xi \in \overline{X}_A$. We say that ξ is *represented by* $\{a_n \in A_n\}_{n\in\mathbb{N}^0}$, and we will write $\xi = \mathfrak{Rep}\left(\{a_n \in A_n\}_{n\in\mathbb{N}^0}\right)$. We say that the sequence $\{a_n \in A_n\}_{n\in\mathbb{N}^0}$ is a *representative* of ξ.

4.24. Let $\Lambda = \{a_n \in A_n\}_{n\in\mathbb{N}^0}$ be a coherent sequence. For any $N > 0$ and $b_N \in A_N$ we will define a coherent sequence $b_N\Lambda = \{c_n \in A_n\}_{n\in\mathbb{N}^0}$ given by

$$c_n = \left\{\begin{array}{cc} \sum_{g\in G(A_N|A_n)} \frac{1}{|G(A_N|A_n)|} g\left(b_N a_n\right) & n < N \\ b_N a_n & n \geq N \end{array}\right. .$$

whence there is the left action of A_N on $\overline{X}_A$ arising from the action of A_N on coherent sequences. Similarly there is the right action of A_N on $\overline{X}_A$. Let $\mathcal{K}\left(\overline{X}_A\right)$ be a C^*-algebra of compact operators with left action on X_A. For any $N \in \mathbb{N}$ the left (resp. right) action of A_N on $\overline{X}_A$ induces the left (resp. right) action of A_N on $\mathcal{K}\left(\overline{X}_A\right)$. If $A \to B(H)$ is a faithful representation then $\overline{X}_A \otimes_A H$ is a pre-Hilbert space with the scalar product given by

$$(\xi \otimes x, \eta \otimes y) = \left(x, \langle\xi, \eta\rangle_{\overline{X}_A} y\right).$$

If $\overline{H}$ is the Hilbert completion of $\overline{X}_A \otimes H$ then $\overline{H}$ is a Hilbert space. For any $n \in \mathbb{N}$ there is the action of A_n on $\overline{H}$ arising from the action of A_n on $\overline{X}_A$. Similarly $\mathcal{K}\left(\overline{X}_A\right)$ acts on $\overline{H}$ and there are natural inclusions $\bigcup_{n\in\mathbb{N}} A_n \subset B\left(\overline{H}\right)$ and $\mathcal{K}\left(\overline{X}_A\right) \subset B\left(\overline{H}\right)$. Moreover there is the natural inclusion of enveloping W^*-algebra $\left(\bigcup_{n\in\mathbb{N}} A_n\right)'' \subset B\left(\overline{H}\right)$. There is a C^*-algebra $\overline{A}$ given by

$$\overline{A} = \mathcal{K}\left(\overline{X}_A\right) \bigcap \left(\bigcup_{n\in\mathbb{N}} A_n\right)'' \subset B\left(\overline{H}\right)$$

There is the natural left (resp. right) action of A_n on both $\mathcal{K}\left(\overline{X}_A\right)$ and $\left(\bigcup_{n\in\mathbb{N}} A_n\right)''$, so there is the left (resp. right) action of A_n of $\overline{A}$. For any $n \in \mathbb{N}$ there is the natural action

of $\overline{G}$ on A_n given by $\overline{g}a = h_n(g)a_n$, which induces natural actions of $\overline{G}$ on both $\overline{X}_A$ and $\overline{A}$, such that $g(a\xi) = (ga)(g\xi)$ for any $a \in \overline{A}$ and $\xi \in \overline{X}_A$. According to the Zorn's lemma [35] there is a maximal irreducible subalgebra $\widetilde{A} \subset \overline{A}$. Let $G \subset \overline{G}$ is a maximal subgroup such that $G\widetilde{A} = \widetilde{A}$. If $g \notin G$ then $\widetilde{A} \bigcap g\widetilde{A} = \{0\}$. From $g^{-1}Gg = G$ it follows that G is a normal subgroup and for any $n \in \mathbb{N}$ there is a homomorphism $h_n|_G : G \to G_n$.

Definition 4.25. The sequence (52) is said to be *faithful* if for any $n \in \mathbb{N}$ following conditions hold:

(a) For any $n \in \mathbb{N}$ the restriction $h_n|_G$ is a group epimorphism, i.e. $h_n(G) = G_n$.

(b) The natural left and right actions of A_n on $\overline{A}$ are faithful.

Lemma 4.26. *If the sequence* (2.1) *is faithful and* $J = \overline{\overline{G}/G}$ *is a set of representatives of* $\overline{G}/G$ *then*

(a) $\overline{A} = \bigoplus_{\overline{g} \in J} \overline{g}\widetilde{A}$*,*

(b) Left and right actions of A_n *on* $\widetilde{A}$ *are faithful for any* $n \in \mathbb{N}$*.*

Proof. a) The algebra $\overline{A}$ is invariant with respect to the $\overline{G}$-action, i.e. $\overline{g}\,\overline{a} \in \overline{A}$ for any $\overline{g} \in \overline{G}$ and $\overline{a} \in \overline{A}$, or $\overline{G}\,\overline{A} = \overline{A}$. If $g \in G$ then $g\widetilde{A} = \widetilde{A}$ for any maximal irreducible subalgebra $\widetilde{A} \subset \overline{A}$. Otherwise if $\overline{g} \in \overline{G} \backslash G$ then $\overline{g}$ transposes irreducible subalgebras, so $\overline{A} = \bigoplus_{\overline{g} \in J} \overline{g}\widetilde{A}$.
b) Consider the right action of A_n on $\overline{A}$. Let $\mathcal{I} \subset A_n$ be the annulator of X_A, i.e. $\mathcal{I}$ is the maximal ideal such that $\widetilde{A}\mathcal{I} = \{0\}$. Since action of A_n on $\overline{A} = \bigoplus_{\overline{g} \in J} \overline{g}\widetilde{A}$ is faithful we have $\bigcap_{\overline{g} \in J} \mathcal{I}^{\overline{g}} = \{0\}$, where $\mathcal{I}^{\overline{g}} = h_n(\overline{g})\,\mathcal{I}$ is the annulator of $\overline{g}\widetilde{A}$. However since $h_n(G) = G_n$ an element $h_n(\overline{g})$ is trivial for any $\overline{g} \in J$ and $\mathcal{I}^{\overline{g}} = h_n(\overline{g})\,\mathcal{I} = \mathcal{I}$, whence $\mathcal{I} = \bigcap_{\overline{g} \in J} \mathcal{I}^{\overline{g}} = \{0\}$ and the action of A_n on $\widetilde{A}$ is faithful. Similarly we can proof that the left action of A_n on $\widetilde{A}$ is faithful. ∎

Definition 4.27. Let (52) be a composable faithful sequence of irreducible C^*-algebras. A Hilbert A-module $\overline{X}_A$ is said to be the *disconnected module* of the sequence (52). The $\overline{G}$ is said to be a *disconnected group* of the sequence (52). The algebra $\overline{A} = \mathcal{K}\left(\overline{X}_A\right) \bigcap \left(\bigcup_{n \in \mathbb{N}} A_n\right)''$ is said to be the *disconnected covering algebra* of the sequence (52). If $\widetilde{A} \subset \overline{A}$ is a maximal irreducible subalgebra and $X_A = \widetilde{A} \otimes_{\overline{A}} \overline{X}_A$, then $\widetilde{A}$ is said to be a *connected covering algebra* of the sequence (52) and $X_A \subset \overline{X}_A$ is said to be a *connected module* of sequence (52). A maximal subgroup $G \subset \overline{G}$ such that $G\widetilde{A} = \widetilde{A}$ (or $GX_A = X_A$) is said a *covering transformation group* of the sequence (52). The group G is a normal subgroup of $\overline{G}$. X_A is a $\widetilde{A}$-A correspondence, i.e. $X_A =_{\widetilde{A}} X_A$. The quadruple $\left(A, \widetilde{A}, _{\widetilde{A}} X_A, G\right)$ is said to be a *noncommutative infinite covering projection* of the sequence (52). A is said to be the *base algebra of the sequence* (52).

Remark 4.28. From the Lemma 4.26 all irreducible subalgebras of $\overline{A}$ are isomorphic. Similarly we can say about G and $_{\widetilde{A}}X_A$. So $\widetilde{A}$, G and $_{\widetilde{A}}X_A$ from the Definition 4.27 are unique up to isomorphisms.

Lemma 4.29. *Let $\Lambda = \{e_n \in A_n\}_{n\in\mathbb{N}^0}$, $\Lambda' = \{e'_n \in A_n\}_{n\in\mathbb{N}^0}$ be coherent sequences such that*

$$e_n = \sum_{g\in G(A_{n+1}|A_n)} g e_{n+1};\; e'_n = \sum_{g\in G(A_{n+1}|A_n)} g e'_{n+1};\; e'_n e^*_n = \sum_{g\in G(A_{n+1}|A_n)} g e'_{n+1} e^*_{n+1}.$$

If $\xi = \mathfrak{Rep}(\Lambda)$, $\xi' = \mathfrak{Rep}(\Lambda')$ then following series

$$\widetilde{a} = \sum_{g\in G} g\xi'\rangle\langle g\xi$$

is strictly convergent and $\langle \eta e'_0 e^*_0, \zeta\rangle_{\overline{X}_A} = \langle \eta, \widetilde{a}\zeta\rangle_{\overline{X}_A}$ for any $\eta, \zeta \in \overline{X}_A$.

Proof. Let $\left\{G^k \subset \overline{G}\right\}_{k\in\mathbb{N}}$ be a $\overline{G}$-covering of the sequence

$$G\left(A_1, A\right) \leftarrow G\left(A_2, A\right) \leftarrow ...$$

where $\overline{G} = \varprojlim G\left(A_n, A\right)$. If $\eta, \zeta \in \overline{X}_A$ are given by $\eta = \mathfrak{Rep}\left(\{b_n \in A_n\}_{n\in\mathbb{N}^0}\right)$, $\zeta = \mathfrak{Rep}\left(\{c_n \in A_n\}_{n\in\mathbb{N}^0}\right)$ then from 4.21 it follows that

$$\langle \eta, \xi'\rangle_{\overline{X}_A} = \lim_{n\to\infty} c^*_n e'_n;\; \langle \xi, \zeta\rangle_{\overline{X}_A} = \lim_{n\to\infty} e^*_n b_n;$$

$$\langle \eta, \xi'\rangle_{\overline{X}_A} \langle \xi, \zeta\rangle_{\overline{X}_A} = \langle \eta, \left(\xi'\rangle\langle\xi\right)\zeta\rangle_{\overline{X}_A} = \lim_{n\to\infty} c^*_n e'_n e^*_n b_n;$$

If $a_m \in \mathcal{K}\left(\overline{X}_A\right)$ is given by

$$a_m = \sum_{g\in G^m} g\xi'\rangle\langle g\xi.$$

then

$$\langle \eta, a_m\zeta\rangle_{\overline{X}_A} = \lim_{n\to\infty} c^*_n \left(\sum_{g\in G^m} \left(h_m(g)(e'_m e^*_m)\right)\right) b_n = \lim_{n\to\infty} c^*_n \left(\sum_{g\in G(A_m,A)} g\left(e'_m e^*_m\right)\right) b_n,$$

whence

$$\lim_{m\to\infty} \langle \eta, a_m\zeta\rangle_{\overline{X}_A} = \lim_{m\to\infty}\lim_{n\to\infty} c^*_n \left(\sum_{g\in G(A_m,A)} g\left(e'_m e^*_m\right)\right) b_n =$$

$$= \lim_{n\to\infty} c^*_n e'_0 e^*_0 b_n = \langle \eta e'_0 e^*_0, \zeta\rangle_{\overline{X}_A}.$$

Form the above equation it follows that the sequence $\{a_m\}_{m\in\mathbb{N}^0}$ is strictly convergent as $m\to\infty$ and $\langle \eta, \left(\lim_{m\to\infty} a_m\right)\zeta\rangle_{\overline{X}_A} = \langle \eta e'_0 e^*_0, \zeta\rangle_{\overline{X}_A}$. Otherwise $\lim_{m\to\infty} a_m = \widetilde{a}$ in the sense of the strict convergence. □

Corollary 4.30. *Let I be a finite or countable set and $\xi_\iota = \mathfrak{Rep}\left(\{e_{\iota n} \in A_n\}_{n\in\mathbb{N}^0}\right)$, $\xi'_\iota = \mathfrak{Rep}\left(\{e'_{\iota n} \in A_n\}_{n\in\mathbb{N}^0}\right) \in \overline{X}_A$ satisfy conditions of the Lemma 4.29. If ξ and ξ' satisfy following condition*

$$\sum_\iota e'_{0,\iota} e^*_{0,\iota} = 1_{M(A)},\; \forall n \in \mathbb{N}^0 \tag{59}$$

in sense of strict topology and then

$$\sum_{\iota \in I, g \in \overline{G}} g\xi'_\iota\rangle\langle g\xi_\iota = 1_{M(\mathcal{K}(\overline{X}_A))}$$

in sense of strict topology.

Proof. From lemma 4.29 it follows that for any $\eta, \zeta \in \overline{X}_A$ following condition hold.

$$\left\langle \eta, \left(\sum_{g \in G} g\xi_\iota\rangle\langle g\xi'_\iota \right) \zeta\rangle_{\overline{X}_A} = \langle \eta e^*_{0,\iota} e'_{0,\iota} \ , \ \zeta \right\rangle_{\overline{X}_A} . \tag{60}$$

From follows (59), (60) it follows that for any $\eta, \zeta \in \overline{X}_A$ following condition hold

$$\left\langle \eta, \left(\sum_{\iota \in I, \ g \in \overline{G}} g\xi_\iota\rangle\langle g\xi_\iota \right) \zeta \right\rangle_{\overline{X}_A} = \left\langle \eta \sum_{\iota \in I} e^*_{0\iota} e'_{0,\iota} \ , \ \zeta\rangle_{\overline{X}_A} = \langle \eta, \zeta \right\rangle_{\overline{X}_A} ,$$

i.e.

$$\sum_{\iota \in I, g \in \overline{G}} g\xi'_\iota\rangle\langle g\xi_\iota = 1_{M(\mathcal{K}(\overline{X}_A))}.$$

□

Corollary 4.31. *In the situation of the corollary 4.30 a linear span of* $\{g\xi'_\iota a\}_{g \in \overline{G}, \ \iota \in I, \ a \in A}$ *is a dense subspace of* $\overline{X}_A$.

Proof. Follows from the Corollary 4.30. □

5 Covering projections of spectral triples

5.1. Let $(\mathcal{A}, H, D)$ be a spectral triple. Similarly to [26] we define a representation of $\pi^1 : \mathcal{A} \to B(H^2)$ given by

$$\pi^1(a) = \begin{pmatrix} a & 0 \\ [D, a] & a \end{pmatrix}.$$

We can inductively construct representations $\pi^s : \mathcal{A} \to B\left(H^{2^s}\right)$ for any $s \in \mathbb{N}$. If π^s is already constructed then $\pi^{s+1} : \mathcal{A} \to B\left(H^{2^{s+1}}\right)$ is given by

$$\pi^{s+1}(a) = \begin{pmatrix} \pi^s(a) & 0 \\ [D, \pi^s(a)] & \pi^s(a) \end{pmatrix} \tag{61}$$

where we assume diagonal action of D on H^{2^s}, i.e.

$$D \begin{pmatrix} x_1 \\ ... \\ x_{2^s} \end{pmatrix} = \begin{pmatrix} Dx_1 \\ ... \\ Dx_{2^s} \end{pmatrix}; \ x_1, ..., x_{2^s} \in H.$$

Definition 5.2. Let

$$\{(\mathcal{A}_n, H_n, D_n)\}_{n\in\mathbb{N}^0} \tag{62}$$

be a sequence of spectral triples and $(\mathcal{A}, H, D) = (\mathcal{A}_0, H_0, D_0)$. The sequence is said to be *coherent* if following conditions hold:

1. There is a sequence of injective *-homomorphisms

$$\mathcal{A} = \mathcal{A}_0 \to \mathcal{A}_1 \to ... \to \mathcal{A}_n \to ... , \tag{63}$$

2. For any $n \in \mathbb{N}$ there is a finite noncommutuative covering projection $(A_{n-1}, A_n, G(A_n, A_{n-1}))$ where A_n is the C^*-completion of $\mathcal{A}_n$ and the *-homomorphism $A_{n-1} \to A_n$ is induced by the inclusion $\mathcal{A}_{n-1} \to \mathcal{A}_n$.

3. The sequence of finite noncommutative covering projections

$$A = A_0 \to A_1 \to ... \to A_n \to ... , \tag{64}$$

   is composable.

4. If $g \in G(A_n, A)$ then $g\mathcal{A}_n = \mathcal{A}_n$, and $\mathcal{A}_n^{G(A_n,A_m)} = \mathcal{A}_m$ for any $m, n \in \mathbb{N}^0$ such that $n > m$.

5. The $\mathcal{A}_n$-module $\mathcal{H}_n^\infty = \bigcap_{k\in\mathbb{N}} \operatorname{Dom} D_n^k$ is given by $\mathcal{H}_n^\infty = \mathcal{A}_n \otimes_{\mathcal{A}} \mathcal{H}^\infty$ where $\mathcal{H}^\infty = \bigcap_{k\in\mathbb{N}} \operatorname{Dom} D^k \subset H$. So $H_n = A_n \otimes H$ and the scalar product on H_n is given by

$$(a \otimes \xi, b \otimes \eta) = (\xi, \langle a, b\rangle_{A_n} \eta)\,;\ \forall a, b \in A_n,\ \forall \xi, \eta \in H_0. \tag{65}$$

   From the above expressions it follows that

   - The space $\mathcal{H}_n^\infty$ is given by $\mathcal{H}_n^\infty = \mathcal{A}_n \otimes_{\mathcal{A}_m} \mathcal{H}_m^\infty$ for any $n > m$,
   - Since $\mathcal{H}_m^\infty = \mathcal{A}_m \otimes_{\mathcal{A}_m} \mathcal{H}_m^\infty$ and $\mathcal{A}_m \subset \mathcal{A}_n$ there is the natural inclusion $\mathcal{H}_m^\infty \subset \mathcal{H}_n^\infty$ and the action of $G(A_n, A_m)$ on $\mathcal{H}_n^\infty$ for any $n > m$.
   - If $g \in G(A_n, A)$ then $g\mathcal{H}_n^\infty = \mathcal{H}_n^\infty$, and $(\mathcal{H}_n^\infty)^{G(A_n,A_m)} = \mathcal{H}_m^\infty$ for any $m, n \in \mathbb{N}^0$ such that $n > m$

6. For any $g \in G(A_n, A)$ and $\xi \in \mathcal{H}_n^\infty$ following conditions hold:

$$g(D_n\xi) = D_n(g\xi)\,;\ \forall \xi \in \mathcal{H}_n^\infty,$$

$$D_n|_{\mathcal{H}_m^\infty} = D_m;\ \forall n > m.$$

Remark 5.3. From the condition 6 of the Definition 5.2 it follows that if $n > m$ then

$$D_n(1_{A_n} \otimes \xi) = 1_{A_n} \otimes D_m\xi;\ \ \forall \xi \in \operatorname{Dom}(D_m)\,;\ \forall n > m.$$

where tensor product means that $H_n = A_n \otimes_{A_m} H_m$. From this property it follows that $\operatorname{Dom}(D_m) \subset \operatorname{Dom}(D_n)$ and $D_n|_{\operatorname{Dom}(D_m)} = D_m$

5.4. Let denote $(\mathcal{A}, D, H) = (\mathcal{A}_0, D_0, H_0)$ and $\mathcal{H}^\infty = \mathcal{H}_0^\infty$. If $\mathcal{H} = A \otimes_{\mathcal{A}} \mathcal{H}^\infty$ then from [2] it follows that $\mathcal{H}$ is a projective finitely generated A-module. From [8] it follows that $\mathcal{H}$ is a finitely generated Hilbert A-module. So $\mathcal{H}_n = A_n \otimes_A \mathcal{H}$ is a finitely generated Hilbert A-module with A valued product given by

$$\langle \xi \otimes a, \eta \otimes b \rangle_{\mathcal{H}_n} = \left\langle \xi, \langle a, b \rangle_{A_n} \eta \right\rangle_{\mathcal{H}}$$

5.5. On the algebraic tensor product $\overline{X}_A \otimes_A H$ there is a $\mathbb{C}$-valued product $(\cdot, \cdot)$ given by

$$(\mu \otimes \xi, \nu \otimes \eta) = \left(\xi, \langle \mu, \nu \rangle_{\overline{X}_A} \eta\right). \tag{66}$$

Denote by $\overline{H}$ the Hilbert completion of the $\overline{X}_A \otimes_A H$ and denote by $\widetilde{H}$ the Hilbert completion of ${}_{\widetilde{A}}X_A \otimes_A H$. From ${}_{\widetilde{A}}X_A \subset \overline{X}_A$ it follows the inclusion $\widetilde{H} \subset \overline{H}$.

Definition 5.6. A sequence $\{\xi_n \in H_n\}_{n \in \mathbb{N}^0}$ is said to be *coherent* in $\overline{H}$ (or $\overline{H}$*-coherent*) if following conditions hold:

1. $\xi_n = \sum_{g \in G(A_{n+1} \mid A_n)} g\xi_{n+1}$
2. The sequence $\{(\xi_n, \xi_n) \in \mathbb{R}\}_{n \in \mathbb{N}}$ is convergent.

5.7. If $\{a_n \in A_n\}_{n \in \mathbb{N}^0}$ is a coherent sequence then for any $\xi \in H$ the sequence $\{a_n \otimes \xi \in H_n\}_{n \in \mathbb{N}^0}$ is coherent in $\overline{H}$. So any $\overline{H}$-coherent sequence $\{\xi_n \in H_n\}_{n \in \mathbb{N}^0}$ corresponds to a functional on $\overline{X}_A \otimes_A H$ given by

$$\{a_n \otimes \xi\} \mapsto \lim_{n \to \infty} (a_n \otimes \xi, \xi_n).$$

The functional can be uniquely extended to $\overline{H}$ and from the Riesz representation theorem it follows the existence of the unique $\overline{\xi} \in \widetilde{H}$ which corresponds to the functional.

Definition 5.8. In the situation 5.7 we say that $\overline{\xi}$ is $\overline{H}$*-represented* by the sequence $\{\xi_n\}$ and we will write $\overline{\xi} = \mathfrak{Rep}_H(\{\xi_n\})$.

5.9. Now we would like to define the unbounded Dirac operator $\widetilde{D}$ on $\widetilde{H}$. It is naturally to define $\widetilde{D}$ on coherent sequences $\{\xi_n \in H_n\}_{n \in \mathbb{N}^0}$ such that

$$\widetilde{D}\, \mathfrak{Rep}_H(\{\xi_n\}) = \mathfrak{Rep}_H(\{D_n \xi_n\}).$$

and then obtain closure of this operator. But the sequence $\{D_n \xi_n\}$ should not be always coherent, and the general definition of $\widetilde{D}$ can be very difficult. However the situation can be simplified in the special case of local covering projections which are described below.

5.10. Let $(\mathcal{A}, H, D)$ (resp. $\left(\widetilde{\mathcal{A}}, \widetilde{H}, \widetilde{D}\right)$) be a spectral triple, let A (resp. $\widetilde{A}$) be the C^*-completion of $\mathcal{A}$ (resp. $\widetilde{\mathcal{A}}$). Suppose that there is a finite noncommutative covering projection $\left(A, \widetilde{A}, G\right)$ such that $G\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}$. Suppose that there are a subspace $\widehat{H} \subset \widetilde{H}$ such that $\widetilde{H} = \bigoplus_{g \in G} g\widehat{H}$, and there is an isomorphism of Hilbert spaces $\varphi : \widehat{H} \to H$ given by $\xi \mapsto \sum_{g \in G} g\xi$.

Definition 5.11. Let us consider the situation 5.10, and let $\mathfrak{A} = (\mathcal{A}, H, D)$, $\widetilde{\mathfrak{A}} = \left(\widetilde{\mathcal{A}}, \widetilde{H}, \widetilde{D}\right)$, $\mathfrak{B} = \left(A, \widetilde{A}, G\right)$. The triple $\left(\mathfrak{A}, \widetilde{\mathfrak{A}}, \mathfrak{B}\right)$ is said to be a *local covering projection of spectral triples* if following conditions hold:

(a) $\mathrm{Dom}\ \widetilde{D} \bigcap \widehat{H} = \varphi^{-1}\left(\mathrm{Dom}\ D\right)$.

(b) $\widetilde{D}\left(\mathrm{Dom}\ \widetilde{D} \bigcap \widehat{H}\right) \subset \widehat{H}$.

(c) $D\left(\varphi\left(\xi\right)\right) = \varphi\left(\widetilde{D}\xi\right)$ for any $\xi \in \widehat{H} \bigcap \mathrm{Dom}\ D$.

Remark 5.12. The meaning of the "local" term is explained in the Remark 6.23.

Lemma 5.13. *If* $\left(\mathfrak{A}, \widetilde{\mathfrak{A}}, \mathfrak{B}\right)$ *is local covering projection of spectral triples then*

$$⨍ \widetilde{D} = |G| ⨍ D.$$

Proof. Let $\widehat{D} = \widetilde{D}|_{\widehat{H}}$. Operator $\widehat{D}$ can be regarded as operator $\mathrm{Dom}\ \widetilde{D} \to \widetilde{H}$ and as $\widehat{H} \bigcap \mathrm{Dom}\ \widetilde{D} \to \widehat{H}$. From the diagram

$$\begin{tikzcd}
\widehat{H} \bigcap \mathrm{Dom}\ \widetilde{D} \arrow[r, "\widehat{D}"] \arrow[d, "\varphi"'] & \widehat{H} \arrow[d, "\varphi"] \\
\mathrm{Dom}\ D \arrow[r, "D"] & H
\end{tikzcd}$$

it follows that operator $\widehat{D}$ is measurable and $⨍ \widehat{D} = ⨍ D$. If $g\widehat{D}$ is given by $\xi \mapsto g\widehat{D}g^{-1}\xi$ then for any $g \in G$ we have $⨍ \widehat{D} = ⨍ g\widehat{D}$. From $\widetilde{D} = \sum_{g \in G} g\widehat{D}$ it follows that

$$⨍ \widetilde{D} = ⨍ \sum_{g \in G} g\widehat{D} = \sum_{g \in G} ⨍ g\widehat{D} = |G| ⨍ D.$$

□

Remark 5.14. It is well known that if $\widetilde{M} \to M$ is an m-fold covering of Riemannian manifold then

$$\int_{\widetilde{M}} \sqrt{\mathrm{det} g(x)} dx^1 \wedge ... \wedge dx^n = m \int_M \sqrt{\mathrm{det} g(x)} dx^1 \wedge ... \wedge dx^n \tag{67}$$

Otherwise from the example 2.21 that in the noncommutative case the noncommutative integral of the Dirac operator $\not{D}$ is proportional to $\int_M \sqrt{\mathrm{det} g(x)} dx^1 \wedge ... \wedge dx^n$. Thus the Lemma 5 is the noncommutative generalization of the Equation (67).

5.15. Suppose that the coherent sequence of spectral triples (62) is such that if $\mathfrak{A} = (\mathcal{A}, H, D)$, $\mathfrak{A}_n = (\mathcal{A}_n, H_n, D_n)$ and $\mathfrak{B}_n = (A, A_n, G(A_n|A))$ then the triple $(\mathfrak{A}, \mathfrak{A}_n, \mathfrak{B}_n)$ is a local covering projection of spectral triples for any $n \in \mathbb{N}$. Let $\widehat{H}^n$ be such that $H^n = \bigoplus_{g \in G(A_n, A)} g\widehat{H}^n$ and $\varphi^n : \widehat{H}^n \to H$ is the isomorphism given by $\xi \mapsto \sum_{g \in G(A_n, A)} g\xi$.

Definition 5.16. Let us consider the situation 5.15. The coherent sequence of spectral triples (62) is said to be *local* if for any $n \in \mathbb{N}$ and $\xi \in \widehat{H}^n$ following conditions hold:

(a)
$$\sum_{g \in G(A_n|A_{n-1})} g\xi \in \widehat{H}^{n-1}.$$

(b) If $\{\xi_n \in H_n\}_{n\in\mathbb{N}}$ is such that $\xi_n \in \widehat{H}^n$ then $\mathfrak{Rep}_{\overline{H}}\left(\{\xi_n \in H_n\}\right) \in \widetilde{H}$.

5.17. If coherent sequence of spectral triples (62) is local then for any $n \in \mathbb{N}$ there is the natural isomorphism $\psi_n : \widehat{H}^n \to \widehat{H}^{n-1}$ given by

$$\psi_n(\xi) = \sum_{g \in G(A_n|A_{n-1})} g\xi$$

and a following condition holds

$$\varphi_n = \psi_1 \circ ... \circ \psi_n.$$

An $\overline{H}$-coherent sequence $\{\xi_n = H_n\}$ is said to be *special* if $\xi_n \in \widehat{H}^n$ for any $n \in \mathbb{N}$. If the sequence is special then there is $\xi \in H$ such that $\xi_n = \varphi_n^{-1}(\xi)$. If Ξ is the set of special sequences then denote by $\widehat{\widehat{H}}$ the Hilbert completion of the $\mathbb{C}$-linear span of $\mathfrak{Rep}_{\overline{H}}(\Xi)$. There is the unique isomorphism $\widehat{\varphi} : \widehat{\widehat{H}} \to H$ which is the extension of the given by $\mathfrak{Rep}_{\overline{H}}\left(\{\xi_n\}\right) \mapsto \xi_0$ map.

Definition 5.18. Suppose that the coherent sequence (62) is local. A $\overline{H}$-coherent sequence $\{\xi_n \in H_n\}_{n\in\mathbb{N}^0}$ is said to be *special* if $\xi_n \in \widehat{H}^n$ for any $n \in \mathbb{N}$.

Lemma 5.19. *Suppose that the coherent sequence* (62) *is local. Let* $G_n = G\left(A_n|A\right)$ *and* $\overline{G} = \varprojlim G_n$. *If* $H = H_0$ *is a separable Hilbert space then following condition hold*

$$\overline{H} = \bigoplus_{g \in \overline{G}} g\widehat{\widehat{H}}.$$

Proof. Let $\{e_\iota\}_{\iota\in I} \subset H$ be a countable orthonormal basis of H, and let $\widehat{H}_\iota = \mathbb{C}e_\iota \subset H$ be a generated by e_ι one dimensional subspace. We say that a $\widetilde{H}$-coherent sequence $\{\xi_n\}$ is *ι-special* if $\xi_n \in H_n^\iota = \bigoplus_{g\in G_n} \mathbb{C}e_\iota^n \subset H_n$ where $e_\iota^n = \varphi_n^{-1}(e_\iota)$. Any $\overline{H}$ coherent Λ sequence can be decomposed $\Lambda = \sum_{\iota\in I} \Lambda_\iota$ into ι-special sequences, so $\overline{H}$ can be decomposed $\overline{H} = \bigoplus_{\iota\in I} \overline{H}_\iota$ where $\overline{H}_\iota$ is generated by ι-special $\overline{H}$-coherent sequences. If $\xi, \eta, \zeta \in \overline{H}_\iota$ are given by $\xi = \mathfrak{Rep}_{\overline{H}}\left(\{\xi_n \in H_n\}_{n\in\mathbb{N}^0}\right)$, $\eta = \mathfrak{Rep}_{\overline{H}}\left(\{\eta_n \in H_n\}_{n\in\mathbb{N}^0}\right)$, $\zeta = \mathfrak{Rep}_{\overline{H}}\left(\{\zeta_n \in H_n\}_{n\in\mathbb{N}^0}\right)$ then

$$(\eta, \xi) = \lim_{n\to\infty} (\eta_n, \xi_n);$$

$$(\eta, \xi)(\xi, \zeta) = \langle \eta, (\xi)(\xi)\zeta \rangle_{\overline{X}_A} = \lim_{n\to\infty} (\eta_n, \xi_n)(\xi_n, \zeta_n);$$

If $\left\{G^k \subset \overline{G}\right\}_{k\in\mathbb{N}}$ is a $\overline{G}$-covering of the sequence $\{G_1 \leftarrow G_2 \leftarrow ...\}$, $\overline{e}_\iota = \mathfrak{Rep}_{\overline{H}}\left(\{e^n_\iota\}\right)$ $a_m \in B\left(\overline{H}_\iota\right)$ be given by

$$a_m = \sum_{g\in G^m} g\overline{e}_\iota\,)(\,g\overline{e}_\iota.$$

then

$$(\eta, a_m\zeta) = \lim_{n\to\infty}\left(\eta_n, \left(\sum_{g\in G^m} h_n(g)\overline{e}_\iota\,)(\,h_n(g)\overline{e}_\iota\right)\zeta_n\right)$$

whence

$$\lim_{m\to\infty}\langle\eta, a_m\zeta\rangle_{\overline{X}_A} = \lim_{m\to\infty}\lim_{n\to\infty}\left(\eta_n, \left(\sum_{g\in G^m} h_n(g)\overline{e}_\iota\,)(\,h_n(g)\overline{e}_\iota\right)\zeta_n\right) =$$

$$= \lim_{n\to\infty}\left(\eta_n, \left(\sum_{g\in G^n} h_n(g)\overline{e}_\iota\,)(\,h_n(g)\overline{e}_\iota\right)\zeta_n\right) = \lim_{n\to\infty}\left(\eta_n, \left(\sum_{g\in G_n} ge^n_\iota\,)(\,ge^n_\iota\right)\zeta_n\right) =$$

$$\lim_{n\to\infty}(\eta_n, \zeta_n) = (\eta, \zeta)$$

Form the above equation it follows that the sequence $\{a_m\}_{m\in\mathbb{N}^0}$ is weakly convergent as $m\to\infty$ and $\lim_{m\to\infty} a_m = 1_{B\left(\overline{H}_\iota\right)}$, i.e.

$$\sum_{g\in\overline{G}} g\overline{e}_\iota\,)(\,g\overline{e}_\iota = 1_{B\left(\overline{H}_\iota\right)}.$$

□

So $\{g\overline{e}_\iota\}_{g\in\overline{G}} \subset \overline{H}_\iota$ is an orthonormal basis of $\overline{H}_\iota$. From $\overline{H} = \bigoplus_{\iota\in I}\overline{H}_\iota$ it follows that $\{g\overline{e}_\iota\}_{g\in\overline{G},\,\iota\in I} \subset \overline{H}$ is an orthonormal basis of $\overline{H}_\iota$, therefore

$$\overline{H} = \bigoplus_{g\in\overline{G},\,\iota\in I} g\overline{H}_\iota = \bigoplus_{g\in\overline{G}} g\widehat{\overline{H}}.$$

Corollary 5.20. *If G is a covering transformation group of the sequence* (64) *then* $\widetilde{H} = \bigoplus_{g\in G} g\widehat{\overline{H}}$.

Proof. Follows from the condition (b) of the Definition 5.16 and the Lemma 5.19. □

5.21. If a coherent sequence of spectral triples (62) is local then there is a densely defined unbounded operator $\widetilde{D}$ on $\widetilde{H}$ given by

$$\mathrm{Dom}\,\widetilde{D} = \bigoplus_{g\in G} g\widehat{\varphi}^{-1}\left(\mathrm{Dom}\,D\right)$$

$$\widetilde{D}\left(\sum_{g\in G} g\xi_g\right) = \sum_{g\in G} g\widehat{\varphi}^{-1}\left(D\widehat{\varphi}\left(\xi_g\right)\right)$$

where $\xi_g \in \widehat{\overline{H}}$ for any $g \in G$.

Definition 5.22. If coherent sequence of spectral triples (62) is local then defined in 5.21 operator $\widetilde{D}$ is said to be the *Dirac operator* of the sequence (62).

Theorem 5.23. *[33] Let T be an unbounded symmetric operator on a Hilbert space H. Then the following are equivalent:*

(a) T is self-adjoint,

(b) T is closed and $\ker(T^* \pm i) = \{0\}$*,*

(c) $\operatorname{ran}(T \pm i) = H$*.*

Corollary 5.24. *If coherent sequence of spectral triples* (62) *is local the Dirac operator is self-adjoint.*

Proof. Since Dirac operator D of the spectral triple $(\mathcal{A}, H, D)$ is self-adjoint it satisfies to the Theorem 5.23. Form 5.21 it follows that $\widetilde{D}$ is a direct sum of infinite copies of D. So $\widetilde{D}$ satisfies to the Theorem 5.23. □

5.25. Let $\{a_n \in A_n\}_{n \in \mathbb{N}^0}$ be a coherent sequence such that $a_n \in \mathcal{A}_n$. For any $s, n \in \mathbb{N}$ there is the s-times differentiable representation $\pi^s_n : \mathcal{A}_n \to B\left(H^{2^s}_n\right)$.

Definition 5.26. A sequence $\{a_n\}_{n \in \mathbb{N}}$ such that $a_n \in \mathcal{A}_n$ is said to be *s-times differentiable* if the sequence

$$\left\{ \frac{1}{|G(A_n \mid A_0)|} \sum_{g \in G(A_n \mid A_0)} g\left((\pi^s(a_n))^* \pi^s(a_n)\right) \in B\left(H^{2^s}\right) \right\}_{n \in \mathbb{N}}$$

is norm convergent as $n \to \infty$. For any s-times differentiable coherent sequence we will define a norm $\|\cdot\|_s$ given by

$$\left\| \{a_n \in \mathcal{A}_n\}_{n \in \mathbb{N}} \right\|_s = \sqrt{\lim_{n \to \infty} \frac{1}{|G(A_n \mid A_0)|} \sum_{g \in G(A_n \mid A_0)} g\left((\pi^s(a_n))^* \pi^s(a_n)\right)}.$$

Definition 5.27. A coherent sequence $\{a_n \in \mathcal{A}_n\}_{n \in \mathbb{N}}$ is said to be *smooth* if it is s-times differentiable for any $s \in \mathbb{N}$.

5.28. If both $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$ are smooth coherent sequences then $\langle \mathfrak{Rep}(\{a_n\}), \mathfrak{Rep}(\{b_n\}) \rangle_{X_A} \in \mathcal{A}$. If $\{a_n\}_{n \in \mathbb{N}}$ is a smooth coherent sequence and $b \in \mathcal{A}$ then the coherent sequence $\{a_n\} b$ given by

$$\{a_n\} b = \{a_n b\}$$

is smooth.

Definition 5.29. Let Ξ be a set of all smooth coherent sequences and $X' \subset \overline{X}_A$ be a $\mathbb{C}$-linear span of $\mathfrak{Rep}(\Xi)$. The completion of X' with respect to seminorms $\|\cdot\|_s$ is said to be the *disconnected smooth* module of the coherent sequence $\{(\mathcal{A}_n, D_n, H_n)\}_{n\in\mathbb{N}^0}$. The disconnected smooth module will be denoted by $\overline{X}^\infty_{\mathcal{A}}$. There is the Fréchet topology on $\overline{X}^\infty_{\mathcal{A}}$ induced by seminorms $\|\cdot\|_s$. There is the natural inclusion $\overline{X}^\infty_{\mathcal{A}} \subset \overline{X}_A$. The intersection $\overline{X}^\infty_{\mathcal{A}} \bigcap \ _{\widetilde{A}}X_A$ of the disconnected smooth module and connected module (Definition 4.27) is said to be the *connected smooth module* which will be denoted by $X^\infty_{\mathcal{A}}$.

Definition 5.30. An element $\kappa \in \mathcal{K}\left(_{\widetilde{A}}X_A\right)$ is said to be *smooth* if following conditions hold:

1. $\kappa X^\infty_{\mathcal{A}} \subset X^\infty_{\mathcal{A}}$.
2. For any $s \in \mathbb{N}$
$$\|\kappa\|_s = \sup_{\xi\in X^\infty_{\mathcal{A}} \ \& \ \|\xi\|_s=1} \|\kappa\xi\|_s < \infty.$$

   Denote by $\mathcal{K}'$ the algebra of smooth operators.

5.31. The $\mathcal{K}'$ is a Fréchet algebra induced by seminorms $\|\cdot\|_s$. From 5.28 it follows that if $\xi, \eta \in X^\infty_{\mathcal{A}}$ then $\xi\rangle\langle\eta \in \mathcal{K}'$

Definition 5.32. If $\xi, \eta \in X^\infty_{\mathcal{A}}$ then the operator $\xi\rangle\langle\eta$ is said to be a *rank-one smooth*. The completion in the Fréchet topology of the linear span of rank-one smooth operators is said to be the *smoothly compact subalgebra*. Denote by $\mathcal{K}^\infty\left(X^\infty_{\mathcal{A}}\right) \subset \mathcal{K}\left(_{\widetilde{A}}X_A\right)$ the smoothly compact subalgebra.

Definition 5.33. If $\left(A, \widetilde{A}, _{\widetilde{A}}X_A, G\right)$ is a noncommutative infinite covering projection of the sequence (64) then from the definition 4.27 it follows that $\widetilde{A} \subset \mathcal{K}\left(_{\widetilde{A}}X_A\right)$. The algebra $\widetilde{\mathcal{A}} = \mathcal{K}^\infty\left(X^\infty_{\mathcal{A}}\right) \bigcap \widetilde{A}$ is said to be the *smooth covering algebra*. The algebra $\widetilde{\mathcal{A}}$ is a Fréchet algebra with a topology induced by seminorms $\|\cdot\|_s$.

Definition 5.34. A local coherent sequence $\{(\mathcal{A}_n, D_n, H_n)\}_{n\in\mathbb{N}^0}$ of spectral triples is said to be *regular* if $\widetilde{\mathcal{A}}$ is a dense subalgebra of $\widetilde{A}$ with respect to C^*-norm of $\widetilde{A}$.

Definition 5.35. If $\{(\mathcal{A}_n, D_n, H_n)\}_{n\in\mathbb{N}^0}$ be a regular local coherent sequence spectral triples then the triple $\left(\widetilde{\mathcal{A}}, \widetilde{H}, \widetilde{D}\right)$ is said to be the *inverse limit* of $\{(\mathcal{A}_n, D_n, H_n)\}_{n\in\mathbb{N}^0}$.

Remark 5.36. The inverse limit from the Definition 5.35 is not an inverse limit in the strict sense of the category theory, it rather looks like an inverse limit.

6 Covering projections of commutative spectral triples

6.1 Covering projections of topological spaces

This section supplies a purely algebraic analog of the topological construction given by the Section 3. Let

$$\mathcal{X} = \mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ... \tag{68}$$

be a sequence of finitely listed regular covering projections such that $\mathcal{X}$ is a locally compact second-countable Hausdorff topological space, and $\mathcal{X}_n$ is connected for any $n \in \mathbb{N}$. Let $\{G_n = G(\mathcal{X}_n|\mathcal{X})\}_{n\in\mathbb{N}}$ be the set of groups of covering transformations. Let $\overline{\mathcal{X}}$ (resp. $\overline{G}$) be a space (resp. a group) described in the Section 3. There is a natural (disconnected) covering projection $\overline{\pi} : \overline{\mathcal{X}} \to \mathcal{X}$. Let $\widetilde{\mathcal{X}} \subset \overline{\mathcal{X}}$ be a connected component. According to the Section 3 there is a normal subgroup $G \subset \overline{G}$ such that $G\widetilde{\mathcal{X}} = \widetilde{\mathcal{X}}$ and a subset $J \subset \overline{G}$ of $\overline{G}/G$ representatives such that $\overline{\mathcal{X}} = \bigsqcup_{g\in J} g\widetilde{\mathcal{X}}$. The restriction $\widetilde{\pi} = \overline{\pi}|_{\widetilde{\mathcal{X}}}$ is a regular covering projection $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X}$ and $\mathcal{X} \approx \widetilde{\mathcal{X}}/G$.

Definition 6.1. Let $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a regular topological covering projection such that the group $G = G\left(\widetilde{\mathcal{X}}|\mathcal{X}\right)$ of covering transformations is finite or countable. Then there is a Hilbert $C_0(\mathcal{X})$-module

$$\mathscr{L}^2\left(\widetilde{\mathcal{X}}_{\mathcal{X}}\right) = \left\{\varphi \in C_b(\widetilde{\mathcal{X}}) \mid \text{if } \phi(x) = \sum_{\widetilde{x}\in\widetilde{\pi}^{-1}(x)} \varphi^*(\widetilde{x})\varphi(\widetilde{x}) \text{ then } \phi \in C_0(\mathcal{X})\right\}. \quad (69)$$

with a $C_0(\mathcal{X})$-valued sesquilinear product given by

$$\langle \xi, \eta \rangle_{\mathscr{L}^2(\widetilde{\mathcal{X}}_{\mathcal{X}})}(x) = \sum_{\widetilde{x}\in\widetilde{\pi}^{-1}(x)} \xi^*(\widetilde{x})\eta(\widetilde{x}). \quad (70)$$

We say that $\mathscr{L}^2\left(\widetilde{\mathcal{X}}_{\mathcal{X}}\right)$ is an *associated with* $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ Hilbert $C_0(\mathcal{X})$-module.

Remark 6.2. If G is a finite group then $\mathscr{L}^2\left(\widetilde{\mathcal{X}}_{\mathcal{X}}\right) \approx C_0\left(\widetilde{\mathcal{X}}\right)_{C_0(\mathcal{X})}$ as Hilbert $C_0(\mathcal{X})$-modules, so this definition compiles with the Theorem 1.17 and with the Equation (38).

Definition 6.3. A C^*-algebra $C_0(\mathcal{X})$ is given by following equation

$$C_0(\mathcal{X}) = \{\varphi \in C_b(\mathcal{X}) \mid \forall\varepsilon > 0 \;\; \exists K \subset \mathcal{X} \; (K \text{ is compact}) \; \& \; \forall x \in \mathcal{X}\backslash K \; |\varphi(x)| < \varepsilon\}.$$

Lemma 6.4. *Let $\mathcal{X}$ be a second-countable compact Hausdorff space. If $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ is a regular covering projection such that $G = G\left(\widetilde{\mathcal{X}} \mid \mathcal{X}\right)$ is countable then $\mathscr{L}^2\left(\widetilde{\mathcal{X}}_{\mathcal{X}}\right) \subset C_0\left(\widetilde{\mathcal{X}}\right)$.*

Proof. Let $\left\{\widetilde{\mathcal{U}}_\iota \subset \widetilde{\mathcal{X}}\right\}_{\iota\in I}$ be a basis of the fundamental covering of $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X}$. Since $\mathcal{X}$ is compact we can select finite family $\left\{\widetilde{\mathcal{U}}_\iota \subset \widetilde{\mathcal{X}}\right\}_{\iota\in I}$, i.e. $\left\{\widetilde{\mathcal{U}}_\iota\right\}_{\iota\in I} = \left\{\widetilde{\mathcal{U}}_1, ..., \widetilde{\mathcal{U}}_n\right\}$. Let

$$G_1 \leftarrow G_2 \leftarrow ...$$

be a coherent sequence of finite groups with epimorphisms $h_i : G \to G_i$, and let $\left\{G^k \subset G\right\}_{k\in\mathbb{N}}$ be a G-covering (See the Definition 1.2). If $\widetilde{\mathcal{V}} = \bigcup_{i=1,...,n} \widetilde{\mathcal{U}}_i$ and $K = \mathfrak{cl}\left(\widetilde{\mathcal{V}}\right)$ is the closure of $\widetilde{\mathcal{V}}$ then K is compact. For any $k \in \mathbb{N}$ the set $K_k = G^k K$ is a finite union of compact sets, whence K_k is compact for any $k \in \mathbb{N}$. If $\widetilde{\mathcal{V}}_k = \bigcup_{k\in\mathbb{N}} G^k\widetilde{\mathcal{V}}$ then from definitions it follows

that $\widetilde{\mathcal{X}} = \bigcup_{k\in\mathbb{N}} \widetilde{\mathcal{V}}_k$. Let $\varphi \in \mathscr{L}^2\left(\widetilde{\mathcal{X}}_{\mathcal{X}}\right)$ be such that $\varphi \notin C_0\left(\mathcal{X}\right)$. From the Definition 6.3 it follows that there is $\varepsilon > 0$ such that for any compact set $K \subset \widetilde{\mathcal{X}}$ there is $\widetilde{x} \in \widetilde{\mathcal{X}} \backslash K$ such that $\left|\varphi\left(\widetilde{x}\right)\right| > \varepsilon$. Let us define a sequence $\left\{\widetilde{x}_i \in \widetilde{\mathcal{X}}\right\}_{i\in\mathbb{N}}$ such that $\left|\varphi\left(\widetilde{x}_i\right)\right| > \varepsilon$ and $\widetilde{x}_i \in \widetilde{\mathcal{X}} \backslash K_i$. There is a sequence $\left\{x_i \in \mathcal{X}\right\}_{i\in\mathbb{N}}$ given by $x_i = \pi\left(\widetilde{x}_{i_j}\right)$. Since $\mathcal{X}$ is compact the sequence $\left\{x_i\right\}_{i\in\mathbb{N}}$ contains a convergent subsequence $\left\{x_{i_j}\right\}_{j\in\mathbb{N}}$. Let $x = \lim_{j\to\infty} x_{i_j}$. Let $\widetilde{x} \in \widetilde{\mathcal{X}}$ be such that $\widetilde{\pi}\left(\widetilde{x}\right) = x$ and $\widetilde{x} \in \widetilde{\mathcal{V}}$. From (69) it follows that the series

$$\sum_{g\in G} \left|\varphi\left(g\widetilde{x}\right)\right|^2$$

is convergent, whence there is $r \in \mathbb{N}$ such that

$$\sum_{g\in G\backslash G^r} \left|\varphi\left(g\widetilde{x}\right)\right|^2 < \frac{\varepsilon^2}{2}. \tag{71}$$

If $\widetilde{\mathcal{W}}$ is an open connected neighborhood of $\widetilde{x}$ which is mapped homeomorphicaly onto $\mathcal{W} = \widetilde{\pi}\left(\widetilde{\mathcal{W}}\right)$ and $\widetilde{\mathcal{W}} \subset \widetilde{\mathcal{V}}$ then there is a real continuous function $\psi : \widetilde{\mathcal{W}} \to \mathbb{R}$ given by

$$\psi\left(y\right) = \sum_{g\in G\backslash G^r} \left|\varphi\left(g\widetilde{y}\right)\right|^2; \text{ where } \widetilde{y} \in \widetilde{\mathcal{W}} \text{ and } \widetilde{\pi}\left(\widetilde{y}\right) = y.$$

There is $s \in \mathbb{N}$ such that $i_s > r$ and $x_{i_j} \in \mathcal{W}$ for any $j \geq s$. If $j > s$ then from $\left|\varphi\left(\widetilde{x}_{i_j}\right)\right| > \varepsilon$ and $\widetilde{x}_{i_j} \notin K_r$ it follows that

$$\psi\left(x_{i_j}\right) = \sum_{g\in G\backslash G^r} \left|\varphi\left(g\widetilde{x}'_{i_j}\right)\right|^2 \geq \left|\varphi\left(\widetilde{x}_{i_j}\right)\right|^2 > \varepsilon^2; \text{ where } \widetilde{x}'_{i_j} \in \widetilde{\mathcal{W}} \text{ and } \widetilde{\pi}\left(\widetilde{x}'_{i_j}\right) = x_{i_j}.$$

Since ψ is continuous and $x = \lim_{j\to\infty} x_{i_j}$ we have $\psi\left(x\right) > \varepsilon^2$. This fact contradicts to the equation (71), and the contradiction proves the lemma.

$\square$

Lemma 6.5. *Let $\mathcal{X}$ be a second-countable locally compact Hausdorff space. If $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X}$ is a regular covering projection such that $G = G\left(\widetilde{\mathcal{X}} \mid \mathcal{X}\right)$ is countable then $\mathscr{L}^2\left(\widetilde{\mathcal{X}}_{\mathcal{X}}\right) \subset C_0\left(\widetilde{\mathcal{X}}\right)$*

Proof. If $\varphi \in \mathscr{L}^2\left(\widetilde{\mathcal{X}}_{\mathcal{X}}\right)$ then $\psi\left(x\right) = \sum_{\widetilde{x}\in\widetilde{\pi}^{-1}(x)} \left|\varphi(\widetilde{x})\right|^2 \in C_0\left(\mathcal{X}\right)$. Let $\varepsilon > 0$ be any number. Form the Definition 6.3 it follows that there is a compact set $K \subset \mathcal{X}$ such $\psi\left(\mathcal{X}\backslash K\right) \subset [0, \varepsilon^2]$. From this fact it follows that $\left|\varphi(\widetilde{x})\right| < \varepsilon$ for any $\widetilde{x} \in \widetilde{\mathcal{X}} \backslash \widetilde{\pi}^{-1}\left(K\right)$. If $\widetilde{K} = \widetilde{\pi}^{-1}\left(K\right)$ then the restriction $\varphi|_{\widetilde{K}}$ belongs to $\mathscr{L}^2\left(\widetilde{K}_K\right)$. From the Lemma 6.4 it follows that $\varphi|_{\widetilde{K}} \in C_0\left(\widetilde{K}\right)$, whence there is a compact set $\widetilde{K}_0 \subset \widetilde{K}$ such that $\left|\varphi\left(\widetilde{x}\right)\right| < \varepsilon$ for any $\widetilde{x} \in \widetilde{K} \backslash \widetilde{K}_0$. In result we have $\left|\varphi\left(\widetilde{x}\right)\right| < \varepsilon$ for any $\widetilde{x} \in \widetilde{\mathcal{X}} \backslash \widetilde{K}_0$. From the Definition 6.3 it follows that $\varphi \in C_0\left(\widetilde{\mathcal{X}}\right)$. $\square$

6.6. Let us consider the sequence (68), and let $\overline{\mathcal{U}} \subset \overline{\mathcal{X}}$ be a connected open set which is mapped homeomorphicaly on $\overline{\pi}\left(\overline{\mathcal{U}}\right)$. Let $\overline{\phi} \in C_0\left(\overline{\mathcal{X}}\right)$ be such that $\overline{\pi}\left(\overline{\mathcal{X}}\backslash\overline{\mathcal{U}}\right) = \{0\}$. For any $n \in \mathbb{N}^0$ there are covering projections $\overline{\pi}^n : \overline{\mathcal{X}} \to \mathcal{X}_n$, $\pi^n : \mathcal{X}_n \to \mathcal{X}$ and there is the descent $\phi_n \in C_0\left(\mathcal{X}_n\right)$ of $\overline{\phi}$ (See Definition 3.7). From the Example 4.9 it follows that

$$C_0(\mathcal{X}) = C_0(\mathcal{X}_0) \to ... \to C_0(\mathcal{X}_n) \to \tag{72}$$

is a sequence of noncommutative covering projections. The sequence $\Lambda = \{\phi_n\}_{n\in\mathbb{N}^0}$ satisfies to (55). From

$$\langle\phi_n, \phi_n\rangle_{C_0(\mathcal{X}_n)}(x) = \sum_{x_n\in(\pi^n)^{-1}(x)} (\phi_n)^*(x_n)\,\phi_n(x_n) = \phi_0^*(x)\,\phi_0(x) \in C_0(\mathcal{X}),$$

it follows that the sequence Λ satisfies to (56), so Λ is a coherent sequence.

Definition 6.7. Let us consider the situation 6.6. The coherent sequence $\Lambda = \{\phi_n\}_{n\in\mathbb{N}^0}$ is said to be the *descent* of $\overline{\phi}$, and we will write $\{\phi_n\}_{n\in\mathbb{N}^0} = \mathfrak{Desc}\left(\overline{\phi}\right)$. The element $\xi = \mathfrak{Rep}\left(\mathfrak{Desc}\left(\overline{\phi}\right)\right) \in \overline{X}_{C_0(\mathcal{X})}$ is said to be a *represented* by $\overline{\phi}$, or $\overline{\phi}$ is a *representative* of ξ. We will write $\xi = \mathfrak{Rep}\left(\overline{\phi}\right)$.

6.8. The sequence (72) is composable. There is the noncommutative covering projection $\left(C_0(\mathcal{X}), \widetilde{A}, {}_{\widetilde{A}}X_{C_0(\mathcal{X})}, G\right)$ of the sequence (72). Let $\left\{\widetilde{\mathcal{U}}_\iota \subset \widetilde{\mathcal{X}}\right\}_{\iota\in I}$ be a basis of the fundamental covering, and let

$$1_{C_b\left(\overline{\mathcal{X}}\right)} = \sum_{g\in G\left(\overline{\mathcal{X}}|\mathcal{X}\right)} \sum_{\iota\in I} g\overline{a}_\iota = \sum_{(g,\iota)\in G\left(\overline{\mathcal{X}}|\mathcal{X}\right)\times I} \overline{a}_{(g,\iota)}$$

be a partition of unity is dominated by $\left\{\widetilde{\mathcal{U}}_\iota\right\}_{\iota\in I}$. If $\overline{e}_\iota = \sqrt{\overline{a}_\iota}$ for any $\iota \in I$ then from the Corollary 4.30 it follows that

$$\sum_{\iota\in I, g\in G\left(\overline{\mathcal{X}}|\mathcal{X}\right)} g\mathfrak{Rep}\left(\overline{e}_\iota\right)\rangle\langle g\mathfrak{Rep}\left(\overline{e}_\iota\right) = 1_{M\left(\mathcal{K}\left(\overline{X}_{C_0(\mathcal{X})}\right)\right)}. \tag{73}$$

If $\Xi = \{\mathfrak{Rep}\left(\overline{e}_\iota\right)\}_{\iota\in I}$ then from the Corollary 4.31 it follows that the set $\overline{G}\Xi C_0(\mathcal{X})$ is dense in $\overline{X}_{C_0(\mathcal{X})}$.

Lemma 6.9. *Let $\mathcal{X}$ be a locally compact second-countable Hausdorff topological space, and let*

$$C_0(\mathcal{X}) = C_0(\mathcal{X}_0) \to ... \to C_0(\mathcal{X}_n) \to ... \tag{74}$$

be an infinite sequence of finite noncommutative covering projections arising from the sequence (68) *of connected topological covering projections. If $\overline{X}_{C_0(\mathcal{X})}$ is a disconnected module of the sequence* (74) *then there is a natural isomorphism $\overline{X}_{C_0(\mathcal{X})} \xrightarrow{\approx} \mathscr{L}^2\left(\overline{\mathcal{X}}_{\mathcal{X}}\right)$ of $C_0(\mathcal{X})$-Hilbert modules.*

Proof. Let $\overline{\mathcal{X}}$ be the disconnected covering space of the sequence (68) with the natural covering projection $\overline{\pi} : \overline{\mathcal{X}} \to \mathcal{X}$ and let $\overline{G} = \varprojlim G\left(\mathcal{X}_n|\mathcal{X}\right)$. For any $\overline{x} \in \overline{X}_{C_0(\mathcal{X})}$ we will

define a test function $\left(\overline{x}, \phi^{\overline{x}}_{[1,0]}, \overline{\mathcal{U}}_{\overline{x}}, \overline{\mathcal{V}}_{\overline{x}}\right)$ subordinated to $\overline{\pi} : \overline{\mathcal{X}} \to \mathcal{X}$. For any $\zeta \in \overline{\mathcal{X}}$ there is the $C_0\left(\mathcal{X}\right)$-valued product

$$\varphi^{\overline{x}}_{\zeta} = \left\langle \zeta, \mathfrak{Rep}\left(\phi^{\overline{x}}_{[1,0]}\right)\right\rangle_{\overline{X}_{C_0(\mathcal{X})}} \in C_0\left(\mathcal{X}\right).$$

If $\overline{\varphi}^{\overline{x}}_{\zeta} \in C_0\left(\overline{\mathcal{X}}\right)$ is the $\overline{\mathcal{U}}_{\overline{x}}$ -lift of $\varphi^{\overline{x}}_{\zeta}$ then the family $\left\{\overline{\varphi}^{\overline{x}}_{\zeta}|_{\overline{\mathcal{V}}_{\overline{x}}} : \overline{\mathcal{V}}_{\overline{x}} \to \mathbb{C}\right\}_{\overline{x}\in\overline{\mathcal{X}}}$ is coherent, whence there is the gluing $\varphi_{\zeta} = \mathfrak{Gluing}\left(\left\{\overline{\varphi}^{\overline{x}}_{\zeta}|_{\overline{\mathcal{V}}_{\overline{x}}}\right\}\right) : \overline{\mathcal{X}} \to \mathbb{C}$. Note that φ_{ζ} is bounded because $\left\|\varphi_{\zeta}\right\| = \left\|\zeta\right\|$, i.e. $\varphi_{\zeta} \in C_b\left(\overline{\mathcal{X}}\right)$. So there is a natural $C_0(\mathcal{X})$-linear map $\alpha : \overline{X}_{C_0(\mathcal{X})} \to C_b(\overline{\mathcal{X}})$, given by $\zeta \mapsto \varphi_{\zeta}$. If $\Xi = \left\{\mathfrak{Rep}\left(\overline{e}_{\iota}\right)\right\}_{\iota\in I}$ then from 6.8 it follows that the set $\overline{G}\ \Xi\ C_0(\mathcal{X})$ is dense in $\overline{X}_{C_0(\mathcal{X})}$. So for any nonzero $\zeta \in \overline{X}_{C_0(\mathcal{X})}$ there is a pair $(\iota, g) \in I \times \overline{G}$ such that $\left\langle \zeta, g\mathfrak{Rep}\left(\overline{e}_{\iota}\right)\right\rangle_{\overline{X}_{C_0(\mathcal{X})}} \neq 0$. If $x \in \mathcal{X}$ is such that $\left\langle \zeta, g\mathfrak{Rep}\left(\overline{e}_{\iota}\right)\right\rangle_{\overline{X}_{C_0(\mathcal{X})}}(x) \neq 0$ then there is the unique $\overline{x} \in g\overline{\mathcal{U}}_{\iota}$ such that $\overline{\pi}(\overline{x}) = x$ and $\alpha(\zeta)(\overline{x}) \neq 0$. It means that α is injective. If $\xi, \eta \in \overline{X}_{C_0(\mathcal{X})}$ then from (73) it follows that

$$\left\langle \xi, \eta\right\rangle_{\overline{X}_{C_0(\mathcal{X})}} = \sum_{\iota\in I, g\in\overline{G}} \left\langle \xi, g\mathfrak{Rep}\left(\overline{e}_{\iota}\right)\right\rangle_{\overline{X}_{C_0(\mathcal{X})}} \left\langle g\mathfrak{Rep}\left(\overline{e}_{\iota}\right), \eta\right\rangle_{\overline{X}_{C_0(\mathcal{X})}}$$

From definition of α it follows that if $e_{\iota} \in C_0\left(\mathcal{X}\right)$ is the descent of the $\overline{e}_{\iota}$ for any $\iota \in I$ then

$$\left(\left\langle \xi, g\mathfrak{Rep}\left(\overline{e}_{\iota}\right)\right\rangle_{\overline{X}_{C_0(\mathcal{X})}} \left\langle g\mathfrak{Rep}\left(\overline{e}_{\iota}\right), \eta\right\rangle_{\overline{X}_{C_0(\mathcal{X})}}\right)(x) = \left(\alpha\left(\xi\right)\left(\overline{x}\right)\right)\left(e_{\iota}\left(x\right)\right)^2\left(\alpha\left(\eta\right)\left(\overline{x}\right)\right)^*$$

where $\overline{x} \in \overline{\mathcal{X}}$ is the unique point such that $\overline{x} \in g\overline{\mathcal{U}}_{\iota}$ and $\overline{\pi}\left(\overline{x}\right) = x$. From above equation it follows that for any $\iota \in I$ following condition hold

$$\left(\sum_{g\in\overline{G}} \left\langle \xi, g\mathfrak{Rep}\left(\overline{e}_{\iota}\right)\right\rangle_{\overline{X}_{C_0(\mathcal{X})}} \left\langle g\mathfrak{Rep}\left(\overline{e}_{\iota}\right), \eta\right\rangle_{\overline{X}_{C_0(\mathcal{X})}}\right)\left(\overline{x}\right) = \sum_{\overline{x}\in\overline{\pi}^{-1}(x)} \left(\alpha\left(\xi\right)\left(\overline{x}\right)\right)\left(e_{\iota}\left(x\right)\right)^2\left(\alpha\left(\eta\right)\left(\overline{x}\right)\right)^*.$$

From $\sum_{\iota\in I} e^2_{\iota} = 1_{M(C_0(\mathcal{X}))}$ it follows that

$$\sum_{\iota\in I, g\in\overline{G}} \left\langle \xi, g\mathfrak{Rep}\left(\overline{e}_{\iota}\right)\right\rangle_{\overline{X}_{C_0(\mathcal{X})}} \left\langle g\mathfrak{Rep}\left(\overline{e}_{\iota}\right), \eta\right\rangle_{\overline{X}_{C_0(\mathcal{X})}}\left(\overline{x}\right) = \sum_{\overline{x}\in\overline{\pi}^{-1}(x)} \left(\alpha\left(\xi\right)\left(\overline{x}\right)\right)\left(\alpha\left(\eta\right)\left(\overline{x}\right)\right)^*,$$

or equivalently

$$\left\langle \xi, \eta\right\rangle_{\overline{X}_{C_0(\mathcal{X})}} = \sum_{\overline{x}\in\overline{\pi}^{-1}(x)} \left(\alpha\left(\xi\right)\left(\overline{x}\right)\right)\left(\alpha\left(\eta\right)\left(\overline{x}\right)\right)^*.$$

In fact the above equation coincides with (70) and from $\left\langle \zeta, \zeta\right\rangle_{\overline{X}_{C_0(\mathcal{X})}} \in C_0\left(\mathcal{X}\right)$ it follows that $\alpha\left(\zeta\right)$ satisfies to (69), i.e. $\alpha\left(\zeta\right) \in \mathscr{L}^2\left(\widetilde{\mathcal{X}}_{\mathcal{X}}\right)$. Otherwise if $\varphi \in \mathscr{L}^2\left(\widetilde{\mathcal{X}}_{\mathcal{X}}\right)$ then the series given by

$$\xi_{\varphi} = \sum_{\overline{g}\in\overline{G},\ \iota\in I} \mathfrak{Rep}\left(\varphi\left(g\overline{e}_{\iota}\right)^2\right) \in X_{C_0(\mathcal{X})},$$

is norm convergent. So there is a linear map $\beta : \mathscr{L}^2\left(\widetilde{\mathcal{X}}_{\mathcal{X}}\right) \to \overline{X}_{C_0(\mathcal{X})}$ given by $\varphi \mapsto \xi_\varphi$. It is easy to check that $\alpha \circ \beta = \mathrm{Id}_{\mathscr{L}^2\left(\widetilde{\mathcal{X}}_{\mathcal{X}}\right)}$ and $\beta \circ \alpha = \mathrm{Id}_{\overline{X}_{C_0(\mathcal{X})}}$, so $\alpha : \overline{X}_{C_0(\mathcal{X})} \xrightarrow{\approx} \mathscr{L}^2\left(\widetilde{\mathcal{X}}_{\mathcal{X}}\right)$ is an isomorphism of $C_0\left(\mathcal{X}\right)$-Hilbert modules. □

Theorem 6.10. *Let $\mathcal{X}$ be a locally compact second-countable Hausdorff topological space, and let us consider the infinite sequence* (74) *of finite noncommutative covering projections arising from the sequence* (68) *of connected topological covering projections. Let $G_n = G\left(C_0(\mathcal{X}_n)|C_0(\mathcal{X})\right) = G\left(\mathcal{X}_n|\mathcal{X}\right)$ be covering transformation groups. Let $\left(C_0(\mathcal{X}), \widetilde{A}, {}_{\widetilde{A}}X_{C_0(\widetilde{\mathcal{X}})}, G\right)$ be a noncommutative infinite covering projection of the sequence* (74). *Then there is a regular connected topological covering projection $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X}$ such that following conditions hold:*

(a) $\widetilde{A} = C_0(\widetilde{\mathcal{X}})$ and ${}_{\widetilde{A}}X_{C_0(\mathcal{X})} = \mathscr{L}^2\left(\widetilde{\mathcal{X}}_{\mathcal{X}}\right)$;

(b) $G(\widetilde{\mathcal{X}}|\mathcal{X}) = G$, $\mathcal{X} = \widetilde{\mathcal{X}}/G$;

(c) The sequence (74) *is faithful.*

Proof. a) If $\overline{X}_{C_0(\mathcal{X})}$ is a disconnected module of the sequence (74) then from the Lemma 6.9 it follows that there is a natural isomorphism $\overline{X}_{C_0(\mathcal{X})} \xrightarrow{\approx} \mathscr{L}^2\left(\overline{\mathcal{X}}_{\mathcal{X}}\right)$ of $C_0\left(\mathcal{X}\right)$-Hilbert modules. From definitions it follows that

$$\sum_{g\in\overline{G}}\sum_{\iota\in I} g\left(\overline{e}_\iota \overline{e}_\iota\right) = 1_{M\left(C_0(\overline{\mathcal{X}})\right)},$$

so any $\varphi \in C_0\left(\overline{X}\right)$ can be represented as the following norm convergent series

$$\varphi = \sum_{g\in\overline{G}}\sum_{\iota\in I} \mathfrak{Rep}\left(\left(g\overline{e}_\iota\right)\varphi\right)\rangle\langle\mathfrak{Rep}\left(g\overline{e}_\iota\right) \in \mathcal{K}\left(\overline{X}_{C_0(\mathcal{X})}\right),$$

whence $C_0\left(\overline{\mathcal{X}}\right) \subset \mathcal{K}\left(\overline{X}_{C_0(\mathcal{X})}\right)$. Let $\left\{G^n \subset \overline{G}\right\}_{n\in\mathbb{N}}$ be a $\overline{G}$-covering (See the Definition 1.2.) of the sequence $\left\{G_n = G\left(\mathcal{X}_n|\mathcal{X}\right)\right\}_{n\in\mathbb{N}}$. If $\varphi_n \in C_b\left(\mathcal{X}\right)$ is given by

$$\varphi_n = \sum_{g'\in\overline{\overline{G}/G_n}} g'\left(\sum_{g\in G^n}\sum_{\iota\in I}\left(g\overline{e}_\iota\right)^2\varphi\right)$$

then $\varphi_n \in A_n$, i.e. there is $a_n \in C_0\left(\mathcal{X}_n\right)$ such that $\varphi_n\xi = a_n\xi$ for any $\xi \in X_{C_0(\mathcal{X})}$. Otherwise $\lim_{n\to\infty}\varphi_n = \varphi$ in sense of the pointwise (=weak) convergence, whence $\varphi \in \left(\bigcup_{n\in\mathbb{N}} C_0\left(\mathcal{X}_n\right)\right)''$. So we have $C_0\left(\overline{\mathcal{X}}\right) \subset \mathcal{K}\left(\overline{X}_{C_0(\mathcal{X})}\right) \bigcap \left(\bigcup_{n\in\mathbb{N}} C_0\left(\mathcal{X}_n\right)\right)''$. Let denote $\overline{A} \stackrel{\text{def}}{=} \mathcal{K}\left(\overline{X}_{C_0(\mathcal{X})}\right) \bigcap \left(\bigcup_{n\in\mathbb{N}} C_0\left(\mathcal{X}_n\right)\right)''$. From definitions it follows that $C_0\left(\mathcal{X}_n\right) \subset L^\infty\left(\overline{\mathcal{X}}\right)$, where $L^\infty\left(\overline{\mathcal{X}}\right)$ is the algebra of essentially bounded complex-valued measurable functions. So $\bigcup_{n\in\mathbb{N}^0} C_0\left(\mathcal{X}_n\right) \subset L^\infty\left(\overline{\mathcal{X}}\right)$. Since $L^\infty\left(\overline{\mathcal{X}}\right)$ is weakly closed, we have $\left(\bigcup_{n\in\mathbb{N}^0} C_0\left(\mathcal{X}_n\right)\right)'' \subset L^\infty\left(\overline{\mathcal{X}}\right)$ and $\overline{A} \subset L^\infty\left(\overline{\mathcal{X}}\right)$. Let μ be a measure on $\overline{X}$ such that the natural representation

of $L^\infty\left(\overline{\mathcal{X}}\right)$ on $L^2\left(\overline{\mathcal{X}},\mu\right)$ is faithful. Let us select any $\overline{x}\in\overline{\mathcal{X}}$. If $\left(\overline{x},\phi^{\overline{x}}_{[1,0]},\overline{\mathcal{U}},\overline{\mathcal{V}}\right)$ is a test function subordinated to $\overline{\pi}$, then $\phi^{\overline{x}}_{[1,0]}\in L^2\left(\overline{\mathcal{X}},\mu\right)$. From the definition of test functions it follows that for any $\varphi\in L^\infty\left(\overline{\mathcal{X}}\right)$ following conditions hold

$$\left(\varphi\left(\phi^{\overline{x}}_{[1,0]}\right)^2\right)|_{\overline{\mathcal{V}}}=\varphi|_{\overline{\mathcal{V}}},$$

$$\left(\varphi\left(\phi^{\overline{x}}_{[1,0]}\right)^2\right)\left(\overline{\mathcal{X}}\backslash\overline{\mathcal{U}}\right)=\{0\}.$$

From the Lemma 6.5 it follows that any element $\xi\in\overline{X}_{C_0(\mathcal{X})}\xrightarrow{\approx}\mathscr{L}^2\left(\widetilde{\mathcal{X}}_{\mathcal{X}}\right)$ can be regarded as element of $C_0\left(\overline{\mathcal{X}}\right)$. If $\kappa=\sum_{i=1}^\infty\eta_i\rangle\langle\xi_i\in\mathcal{K}\left(\overline{X}_{C_0(\mathcal{X})}\right)$ is a compact operator then a following series

$$\psi=\left\langle\mathfrak{Rep}\left(\phi^{\overline{x}}_{[1,0]}\right),\kappa\,\mathfrak{Rep}\left(\phi^{\overline{x}}_{[1,0]}\right)\right\rangle_{\overline{X}_{C_0(\mathcal{X})}}=$$

$$=\sum_{i=1}^\infty\left\langle\mathfrak{Rep}\left(\phi^{\overline{x}}_{[1,0]}\right),\eta_i\right\rangle_{\overline{X}_{C_0(\mathcal{X})}}\left\langle\xi_i,\mathfrak{Rep}\left(\phi^{\overline{x}}_{[1,0]}\right)\right\rangle_{\overline{X}_{C_0(\mathcal{X})}}\in C_0\left(\mathcal{X}\right)$$

is norm convergent. If $\overline{\psi}\in C_0\left(\overline{\mathcal{X}}\right)$ is the $\overline{\mathcal{U}}$-lift of ψ then

$$\overline{\psi}=\left(\sum_{i=1}^\infty\eta_i^*\xi_i\right)\left(\phi^{\overline{x}}_{[1,0]}\right)^2\tag{75}$$

where $\xi_i,\eta_i\in\mathscr{L}^2\left(\overline{\mathcal{X}}_{\mathcal{X}}\right)$ are regarded as elements of $C_0\left(\overline{\mathcal{X}}\right)$. Otherwise if $\kappa=\overline{\varphi}\in L^\infty\left(\overline{\mathcal{X}}\right)$ then

$$\psi=\sum_{\overline{g}\in\overline{G}}\overline{g}\left(\overline{\varphi}\left(\phi^{\overline{x}}_{[1,0]}\right)^2\right),$$

whence $\left(\overline{\varphi}\left(\phi^{\overline{x}}_{[1,0]}\right)^2\right)$ is the $\overline{\mathcal{U}}$-lift of ψ. From above equations it follows that

$$\overline{\varphi}\left(\phi^{\overline{x}}_{[1,0]}\right)^2=\overline{\psi}=\left(\sum_{i=1}^\infty\eta_i^*\xi_i\right)\left(\phi^{\overline{x}}_{[1,0]}\right)^2.$$

where $\overline{\psi}\in C_0\left(\overline{\mathcal{X}}\right)$ and the series is norm convergent. From the Definition 3.10 of $\phi^{\overline{x}}_{[1,0]}$ it follows that

$$\overline{\varphi}|_{\overline{\mathcal{V}}}=\overline{\psi}|_{\overline{\mathcal{V}}}.$$

Since $\overline{\psi}$ is a continuous, the function $\overline{\varphi}|_{\overline{\mathcal{V}}}$ is also continuous. Thus any $\overline{x}\in\overline{\mathcal{X}}$ has a neighborhood $\overline{\mathcal{V}}$ such that the restriction $\overline{\varphi}|_{\overline{\mathcal{V}}}$ is continuous, whence $\overline{\varphi}$ is continuous, and $\overline{\varphi}\in C_b\left(\overline{\mathcal{X}}\right)$ because $\|\overline{\varphi}\|=\|\kappa\|$. We have selected an arbitrary point $\overline{x}\in\overline{\mathcal{X}}$, therefore from (75) it follows that

$$\overline{\varphi}\left(\overline{x}\right)=\left(\sum_{i=1}^\infty\eta_i^*\xi_i\right)\left(\overline{x}\right);\ \forall\overline{x}\in\overline{\mathcal{X}}.$$

Note that for any $\kappa \in \mathcal{K}\left(\overline{X}_{C_0(\mathcal{X})}\right)$ following condition hold

$$\|\kappa\| = \sup_{\mu,\nu} \left\| \langle \mu, \kappa\nu \rangle_{\overline{X}_{C_0(\mathcal{X})}} \right\|; \text{ where } \mu, \nu \in \overline{X}_{C_0(\mathcal{X})} \text{ and } \|\mu\| = \|\nu\| = 1.$$

From the definition of the norm it follows that

$$\left\| \overline{\varphi} - \sum_{i=1}^{k} \eta_i^* \xi_i \right\| = \sup_{\overline{x} \in \overline{\mathcal{X}}} \left| \overline{\varphi} - \sum_{i=1}^{k} \eta_i^* \xi_i \right| (\overline{x}).$$

However

$$\left| \overline{\varphi} - \sum_{i=1}^{k} \eta_i^* \xi_i \right| (\overline{x}) \leq \left\| \left\langle \phi_{[1,0]}^{\overline{x}}, \left(\overline{\varphi} - \sum_{i=1}^{k} \eta_i \rangle \langle \xi_i \right) \phi_{[1,0]}^{\overline{x}} \right\rangle \right\|$$
$$\leq \sup_{\alpha,\beta} \left\| \left\langle \mu, \left(\overline{\varphi} - \sum_{i=1}^{k} \eta_i \rangle \langle \xi_i \right) \nu \right\rangle \right\| = \left\| \overline{\varphi} - \sum_{i=1}^{k} \eta_i \rangle \langle \xi_i \right\|;$$
$$\text{where } \mu, \nu \in \overline{X}_{C_0(\mathcal{X})}, \text{ and } \|\mu\| = \|\nu\| = 1,$$

whence

$$\left\| \overline{\varphi} - \sum_{i=1}^{k} \eta_i^* \xi_i \right\| \leq \left\| \overline{\varphi} - \sum_{i=1}^{k} \eta_i \rangle \langle \xi_i \right\|. \tag{76}$$

The series $\sum_{i=1}^{\infty} \eta_i \rangle \langle \xi_i$ is norm convergent, and from (76) it follows that the series $\sum_{i=1}^{\infty} \eta_i^* \xi_i$ is also norm convergent. From the Lemma 6.5 if follows that $\eta_i, \xi_i \in C_0\left(\overline{\mathcal{X}}\right)$ whence $\sum_{i=1}^{k} \eta_i^* \xi_i \in C_0\left(\overline{\mathcal{X}}\right)$. Since the series $\sum_{i=1}^{\infty} \eta_i^* \xi$ is norm convergent and any partial sum $\sum_{i=1}^{k} \eta_i^* \xi$ belongs to $C_0\left(\overline{\mathcal{X}}\right)$ following condition hold

$$\overline{\varphi} = \sum_{i=1}^{\infty} \eta_i^* \xi_i \in C_0\left(\overline{\mathcal{X}}\right)$$

and

$$\mathcal{K}\left(\overline{X}_{C_0(\mathcal{X})}\right) \bigcap \left(\bigcup_{n \in \mathbb{N}} C_0\left(\mathcal{X}_n\right) \right)'' \subset C_0\left(\overline{\mathcal{X}}\right).$$

In result we have

$$\mathcal{K}\left(\overline{X}_{C_0(\mathcal{X})}\right) \bigcap \left(\bigcup_{n \in \mathbb{N}} C_0\left(\mathcal{X}_n\right) \right)'' = C_0\left(\overline{\mathcal{X}}\right).$$

If $\widetilde{\mathcal{X}} \subset \overline{\mathcal{X}}$ is a connected component, then $C_0\left(\widetilde{\mathcal{X}}\right) \subset C_0\left(\overline{\mathcal{X}}\right)$ is a maximal irreducible subalgebra. If $C_0\left(\overline{\mathcal{X}}\right) = C_0\left(\widetilde{\mathcal{X}}\right) \oplus A'$ then $X_{C_0(\mathcal{X})} = \left\{\xi \in \overline{X}_{C_0(\mathcal{X})} \mid A'\xi = \{0\}\right\} = \mathscr{L}^2\left(\widetilde{\mathcal{X}}_{\mathcal{X}}\right)$.
b) The action of $\overline{G}$ on $C_0\left(\overline{\mathcal{X}}\right)$ arises from the action of $\overline{G}$ on $\overline{\mathcal{X}}$, and according to topological construction 25 we have $\mathcal{X} = \overline{\mathcal{X}}/\overline{G}$. If $G \subset \overline{G}$ is the maximal subgroup such that $GC_0\left(\widetilde{\mathcal{X}}\right) = C_0\left(\widetilde{\mathcal{X}}\right)$ then G is the maximal subgroup such that $G\widetilde{\mathcal{X}} = \widetilde{\mathcal{X}}$ and vice versa. So $G(\widetilde{\mathcal{X}}|\mathcal{X}) = G$, $\mathcal{X} = \widetilde{\mathcal{X}}/G$.
c) If $a \in C_0\left(\mathcal{X}_n\right)$ is a nonzero function then there is an open set $\mathcal{U} \subset \mathcal{X}_n$ such that

- $a|_{\mathcal{U}} \neq 0$;
- $\mathcal{U}$ is evenly covered by $\pi_n : \overline{\mathcal{X}} \to \mathcal{X}_n$.

If $\overline{\mathcal{U}} \subset \overline{\mathcal{X}}$ be an connected open subset which is homeomorphically mapped onto $\mathcal{U}$ and $\overline{a} \in C_0\left(\overline{\mathcal{X}}\right)$ is such that $\overline{a}|_{\overline{\mathcal{U}}} \neq 0$ then $\overline{a}a \neq 0$ and $a\overline{a} \neq 0$, so the actions of $C_0\left(\mathcal{X}_n\right)$ on $C_0\left(\overline{\mathcal{X}}\right)$ are faithful. From [35] it follows that the sequence of regular covering projections

$$\widetilde{\mathcal{X}} \to \mathcal{X}_n \to \mathcal{X}$$

induces the epimorphism of groups $G\left(\widetilde{\mathcal{X}}|\mathcal{X}\right) \to G\left(\mathcal{X}_n|\mathcal{X}\right)$. From $G\left(C_0\left(\widetilde{\mathcal{X}}\right)|C_0\left(\mathcal{X}\right)\right) \approx G\left(\widetilde{\mathcal{X}}|\mathcal{X}\right)$ and $G\left(C_0\left(\mathcal{X}_n\right)|C_0\left(\mathcal{X}\right)\right) \approx G\left(\mathcal{X}_n|\mathcal{X}\right)$ follows that the natural map $G\left(C_0\left(\widetilde{\mathcal{X}}\right)|C_0\left(\mathcal{X}\right)\right) \to G\left(C_0\left(\mathcal{X}_n\right)|C_0\left(\mathcal{X}\right)\right)$ is a group epimorphism.

□

Lemma 6.11. *Let $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a regular topological covering projection by a connected space $\widetilde{\mathcal{X}}$ such that a group $G = G\left(\widetilde{\mathcal{X}}|\mathcal{X}\right)$ of covering transformations is countable, i.e. $\mathcal{X} \approx \widetilde{\mathcal{X}}/G$. Let $\ldots \to G_n \to \ldots \to G_1$ a coherent sequence (See Definition 1.2) of finite G-quotients with epimorphisms $h_n : G \to G_n$. Let $\mathcal{X}_n = \widetilde{\mathcal{X}}/\ker h_n$. For any $n \in \mathbb{N}$ there is a natural finitely listed covering projection $\pi_n : \mathcal{X}_n \to \mathcal{X}$ such that $G\left(\mathcal{X}_n|\mathcal{X}\right) = G_n$. There is a following sequence of *-homomorphisms*

$$C_0(\mathcal{X}) \to C_0(\mathcal{X}_1) \to \ldots \to C_0(\mathcal{X}_n) \to \ldots . \tag{77}$$

If $\left(C_0(\mathcal{X}), \widetilde{A}, {}_{C_0\left(\widetilde{\mathcal{X}}'\right)}X_{C_0(\mathcal{X})}, G'\right)$ noncommutative infinite covering projection of the sequence (77) then $\widetilde{\mathcal{X}}' \approx \widetilde{\mathcal{X}}$ and $G' \approx G$.

Proof. If $\overline{\mathcal{X}}$ is a disconnected covering space of the sequence (68), $\overline{G}$ is a disconnected group of (68) and $J = \overline{\overline{G}/G} \subset \overline{G}$, $J' = \overline{\overline{G}/G'}$ are sets of representatives of G and G' in $\overline{G}$ then from

$$\overline{\mathcal{X}} = \bigsqcup_{g \in J} g\widetilde{\mathcal{X}} = \bigsqcup_{g \in J'} g\widetilde{\mathcal{X}}'.$$

it follows that $\widetilde{\mathcal{X}}' \approx \widetilde{\mathcal{X}}$ and $G' \approx G$ because both $\widetilde{\mathcal{X}}'$ and $\widetilde{\mathcal{X}}$ are connected. □

Remark 6.12. From the Theorem 6.10 and the Lemma 6.11 it follows that an infinite covering can be constructed algebraically.

6.2 Covering projections of spectral triples

In this section we will consider following objects:

1. A compact orientable Riemannian manifold (M, ε) without boundary with a Spin$^{\mathbf{c}}$ structure $(\mathcal{S}, C)$ where $\mathcal{S} = \Gamma_{\text{smooth}}(M, S)$ where Γ_{smooth} means the functor of C^∞-sections and S is the spinor bundle (See the Definition 2.10).
2. A covering projection $\widetilde{\pi} : \widetilde{M} \to M$.

3. The pullback $\widetilde{S}$ of S by $\widetilde{\pi}$ (See the Definition 1.22).

6.13. Let $\pi : \widetilde{M} \to M$ be a regular covering projection. From the Proposition 1.37 it follows that $\widetilde{M}$ has the natural structure of C^∞-manifold. Moreover from [9] it follows that $\widetilde{M}$ is a Riemannian manifold with *covering metric* $\widetilde{g}$.

Definition 6.14. A $\mathbb{R}$-linear map $\varphi : \Gamma_{\text{smooth}}(M, S) \to \Gamma_{\text{smooth}}(M, S)$ is said to be *local* if for any open subset U and any $s', s'' \in \Gamma_{\text{smooth}}(M, S)$ such that from $s'|_U = s''|_U$ it follows that

$$\varphi\left(s'\right)|_U = \varphi\left(s''\right)|_U.$$

6.15. Let $\varphi : \Gamma_{\text{smooth}}(M, S) \to \Gamma_{\text{smooth}}(M, S)$ be a local operator and let $\left\{\widetilde{U}_\iota\right\}_{\iota \in I}$ be a one-to-one covering with respect to $\widetilde{\pi}$ and $U_\iota = \widetilde{\pi}\left(\widetilde{U}_\iota\right)$. For any $\iota \in I$ there is a $\mathbb{C}$-linear isomorphism $\alpha_\iota : \Gamma_{\text{smooth}}\left(U_\iota, S|_{U_\iota}\right) \to \Gamma_{\text{smooth}}\left(\widetilde{U}_\iota, \widetilde{S}|_{\widetilde{U}_\iota}\right)$. If $\widetilde{s} \in \Gamma_{\text{smooth}}\left(\widetilde{M}, \widetilde{S}\right)$ then for any $\iota \in I$ there is the unique section $\widetilde{t}_\iota \in \Gamma_{\text{smooth}}\left(\widetilde{U}_\iota, \widetilde{S}|_{\widetilde{U}_\iota}\right)$ given by

$$\widetilde{t}_\iota = \alpha_\iota\left(\varphi\left(\alpha_\iota^{-1}\left(\widetilde{s}|_{\mathcal{U}_\iota}\right)\right)\right).$$

A family $\left\{\widetilde{t}_\iota\right\}_{\iota \in I}$ is coherent because φ is local. So there is the gluing $\widetilde{t} = \mathfrak{Gluing}\left(\left\{\widetilde{t}_\iota\right\}_{\iota \in I}\right) \in \Gamma_{\text{smooth}}\left(\widetilde{M}, \widetilde{S}\right)$. In fact definition of $\widetilde{t}$ does not depend on the family $\left\{\widetilde{U}_\iota\right\}_{\iota \in I}$.

Definition 6.16. In the situation 6.15 there is a local operator $\widetilde{\varphi} : \Gamma_{\text{smooth}}\left(\widetilde{M}, \widetilde{S}\right) \to \Gamma_{\text{smooth}}\left(\widetilde{M}, \widetilde{S}\right)$ given by $\widetilde{s} \mapsto \widetilde{t}$. The operator $\widetilde{\varphi}$ is said to be the *$\widetilde{\pi}$-pullback* of φ. Henceforth we write $\widetilde{\varphi} = \mathfrak{pullback}_{\widetilde{\pi}}(\varphi)$.

Remark 6.17. Any pullback is $G\left(\widetilde{M}|M\right)$-equivariant, i.e.

$$\mathfrak{pullback}_{\widetilde{\pi}}(\varphi)\left(g\widetilde{s}\right) = g\left(\mathfrak{pullback}_{\widetilde{\pi}}(\varphi)\left(\widetilde{s}\right)\right)$$

for any $\widetilde{s} \in \Gamma_{\text{smooth}}\left(\widetilde{M}, \widetilde{S}\right)$ and $g \in G\left(\widetilde{M}|M\right)$.

6.18. Similarly to (16) we can define a scalar product on $\widetilde{H} = \widetilde{A} \otimes_A H$ given by

$$(\widetilde{\phi}, \widetilde{\psi}) = \int_{\widetilde{M}} (\widetilde{\phi} \mid \widetilde{\psi}) \nu_{\widetilde{g}} \quad \text{for } \widetilde{\phi}, \widetilde{\psi} \in \widetilde{\mathcal{S}}$$

where $\nu_{\widetilde{g}}$ means Riemannian measure on $\widetilde{M}$. Since $\widetilde{M} \to M$ is a finitely listed covering above integral can be presented by following way

$$\int_{\widetilde{M}} (a \otimes \phi \mid b \otimes \psi)\, \widetilde{\nu}_g = \int_M \left(\sum_{y = \pi_n^{-1}(x)} a^*(y)\, b(y)\right) \left(\phi(x) \mid \psi(x)\right)\, \nu_g = \left(\phi, \langle a, b \rangle_{C(\widetilde{M})} \psi\right), \tag{78}$$

i.e. the Hilbert scalar product on $\widetilde{H}$. complies with (65) It is clear that both the Dirac $\not{D}$ operator and charge conjugation operator C are local, so there are pullbacks $\widetilde{\not{D}} \overset{\text{def}}{=} \mathfrak{pullback}_{\widetilde{\pi}}(\not{D})$, $\widetilde{C} \overset{\text{def}}{=} \mathfrak{pullback}_{\widetilde{\pi}}(C)$. So $\left(\widetilde{M}, \widetilde{\varepsilon}\right)$ is a orientable Reimannian manifold without boundary with the $\text{Spin}^{\mathbf{c}}$ structure $\left(\widetilde{\mathcal{S}}, \widetilde{C}\right)$ where $\widetilde{\mathcal{S}} = \Gamma_{\text{smooth}}\left(\widetilde{M}, \widetilde{S}\right)$. If $\dim M$ is even then from the Definition 2.33 it follows the existence of the operator Γ which is also local, so there is the pullback $\widetilde{\Gamma} \overset{\text{def}}{=} \mathfrak{pullback}_{\widetilde{\pi}}(\Gamma)$.

Definition 6.19. Let us consider the situation 6.18, and suppose that the covering projection $\widetilde{\pi} : \widetilde{M} \to M$ is finite fold and regular. The spectral triple $\left(C^\infty\left(\widetilde{M}\right), L^2(\widetilde{M}, \widetilde{S}), \widetilde{\not{D}}\right)$ is said to be the $\widetilde{\pi}$-*pullback* of $\left(C^\infty(M), L^2(M,S), \not{D}\right)$. We will write $\left(C^\infty\left(\widetilde{M}\right), L^2(\widetilde{M}, \widetilde{S}), \widetilde{\not{D}}\right) = \mathfrak{pullback}_{\widetilde{\pi}}\left(\left(C^\infty(M), L^2(M,S), \not{D}\right)\right)$.

Remark 6.20. If a sequence $\widetilde{\widetilde{M}} \xrightarrow{\widetilde{\widetilde{\pi}}} \widetilde{M} \xrightarrow{\widetilde{\pi}} M$ is such that both $\widetilde{\widetilde{\pi}}$ and $\widetilde{\pi}$ are covering projections then following condition hold

$$\mathfrak{pullback}_{\widetilde{\widetilde{\pi}}\circ\widetilde{\pi}}\left(\left(C^\infty(M), L^2(M,S), \not{D}\right)\right) = \mathfrak{pullback}_{\widetilde{\widetilde{\pi}}}\left(\mathfrak{pullback}_{\widetilde{\pi}}\left(\left(C^\infty(M), L^2(M,S), \not{D}\right)\right)\right).$$

Lemma 6.21. *Let (M, ε) be a compact orientable Riemannian manifold without boundary with a* $\text{Spin}^{\mathbf{c}}$ *structure $(\mathcal{S}, C)$. Let*

$$M = M_0 \leftarrow M_1 \leftarrow ... \leftarrow M_n \leftarrow ...$$

*be a sequence of finite fold regular covering projections which induces the sequence of *-homomorphisms*

$$C(M) = C(M_0) \to C(M_1) \to ... \to C(M_n) \to ....$$

If $\pi_n : M_n \to M$ is a natural covering projection for and $\left(C^\infty(M_n), L^2(M_n, S_n), \not{D}_n\right)$ is the π_n-pullback of $\left(C^\infty(M), L^2(M,S), \not{D}\right)$ then the sequence $\left\{\left(C^\infty(M_n), L^2(M_n, S_n), \not{D}_n\right)\right\}_{n\in\mathbb{N}^0}$ is a coherent sequence of spectral triples.

Proof. We need check conditions 1-6 of the Definition 5.2.

1. There is a sequence of injective *-homomorphisms

$$C^\infty(M) = C^\infty(M_1) \to C^\infty(M_2) \to ... \to C^\infty(M_n) \to ... .$$

2. For any $n \in \mathbb{N}$ algebra $C(M_n)$ is the C^*-completion of $C^\infty(M_n)$ and there is a finite noncommutuative covering projection $(C(M_{n-1}), C(M_n), G(C(M_n) \mid C(M_{n-1})))$.

3. The sequence of finite noncommutative covering projections

$$C(M) = C(M_1) \to C(M_2) \to ... \to C(M_n) \to ... \quad (79)$$

is composable.

4. If $g \in G(M_n, M)$ then $gC^\infty(M_n) = C^\infty(M_n)$, and $C^\infty(M_n)^{G(M_n|M_m)} = C^\infty(M_m)$ for any $m, n \in \mathbb{N}^0$.

5. The $C^\infty(M_n)$-module $\Gamma_{\text{smooth}}(M_n, S_n) = \bigcap_{k\in\mathbb{N}} \text{Dom}\, \not{D}_n^k$ is given by $\Gamma_{\text{smooth}}(M_n, S_n) = C^\infty(M_n) \otimes_{C^\infty(M)} \Gamma_{\text{smooth}}(M, S)$ where $\Gamma_{\text{smooth}}(M, S) = \bigcap_{k\in\mathbb{N}} \text{Dom}\, \not{D}^k \subset L^2(M, S, \nu^g)$. The Hilbert space $H = L^2(M, S)$ of any commutative spectral triple is a Hilbert completion of the space $\mathcal{S} = \Gamma_{\text{smooth}}(M, S)$ of smooth sections of the spinor bungle. However $\mathcal{S}$ is a dense subspace of $\overline{\mathcal{S}} = \Gamma(M, S)$ continuous sections of the spinor bundle. The bungle S_n is a π_n-lift of S, and from 1.30 it follows that
$$\overline{\mathcal{S}_n} = \Gamma(M_n, S_n) = C(M_n) \otimes_{C(M)} \overline{\mathcal{S}}$$
whence $L^2(M_n, S_n)$ is the Hilbert completion of $C(M_n) \otimes_{C(M)} L^2(M, S)$. However the Hilbert completion of $C(M_n) \otimes_{C(M)} L^2(M, S)$ coincides with $C(M_n) \otimes_{C(M)} L^2(M, S)$ because $C(M_n)$ is a finitely generated $C(M)$-module, i.e. $H_n = C(M_n) \otimes_{C(M)} H_0$. From (78) it follows that for any $n \in \mathbb{N}$ the Hilbert scalar product on H_n complies with (65).

6. According to definition $\not{D}_n = \mathfrak{pullback}_{\pi^n}(\not{D})$ and from the Remark 6.17 it follows that $\not{D}_n$ is $G(M_n|M)$ equivariant, i.e. for any $g \in G(M_n, M)$ and $\xi \in \xi \in \Gamma_{\text{smooth}}(M_n, S_n)$ following condition hold
$$g(\not{D}\xi) = \not{D}(g\xi).$$
From the Definition 6.16 it follows that if $m < n$ then
$$\not{D}_n|_{\Gamma_{\text{smooth}}(M_m, S_m)} = \not{D}_m$$

$\square$

Lemma 6.22. *The coherent sequence of spectral triples from the Lemma 6.21 is local.*

Proof. From the Theorem 6.10 it follows that there are a connected topological space $\widetilde{M}$ and a covering projection $\widetilde{p} : \widetilde{M} \to M$ such that $\left(C(M), C\left(\widetilde{M}\right), {}_{C(\widetilde{M})}X_{C(M)}, G\right)$ is the noncommutative covering projection of the sequence (79). Let $x \in M$ be any point and $\widetilde{x} \in \widetilde{M}$ is such that $\widetilde{p}(\widetilde{x}) = x$. From the Remark 3.5 there is the cut loci $\Omega_x \subset M$ such that following conditions hold:

1. $1_{\Omega_x} \in L^\infty(M)$ and if $1_M = 1_{\Omega_x}$ as element of $L^\infty(M)$,

2. There is a connected set $\widetilde{\Omega}_{\widetilde{x}}$ which is mapped homeomophically on Ω_x and $\widetilde{\Omega}_{\widetilde{x}}$ is a fundamental domain of the $\widetilde{p} : \widetilde{M} \to M$. For any $n \in \mathbb{N}$ there is a covering projection $\widetilde{p}_n : \widetilde{M} \to M_n$ and $\Omega^n_{\widetilde{p}_n(\widetilde{x})} = \widetilde{p}_n\left(\widetilde{\Omega}_{\widetilde{x}}\right)$ is a fundamental domain of the covering projection $p_n : M_n \to M$.

If $\widehat{H}^n = 1_{\Omega^n_{\widetilde{p}_n(\widetilde{x})}} L^2(M_n, S_n)$ then from $1_{M_n} = \sum_{g \in G(M_n|M)} g\, 1_{\Omega^n_{\widetilde{p}_n(\widetilde{x})}}$ it follows that

$$L^2(M_n, S_n) = \bigoplus_{g \in G(M_n|M)} g \widehat{H}^n.$$

From locality of the operators $\not{D}_n$ it follows that they satisfy the Definition 5.11. The condition (a) of the Definition 5.16 follows from

$$1_{\Omega^{n-1}_{\widetilde{p}_{n-1}(\widetilde{x})}} = \sum_{g \in G(M_n|M_{n-1})} g\, 1_{\Omega^n_{\widetilde{p}_n(\widetilde{x})}}$$

The condition (b) of the Definition 5.16 follows from $\widetilde{\Omega}_{\widetilde{x}} \subset \widetilde{M}$. Really if $\left\{\xi_n \in L^2(M_n, S_n)\right\}_{n \in \mathbb{N}}$ is such that $\xi_n \in 1_{\Omega^n_{\widetilde{p}_n(\widetilde{x})}} L^2(M_n, S_n)$ then from ${}_{C(M)}X_{C_0(\widetilde{M})} = C_0\left(\widetilde{M}\right) \otimes_{C_0\left(\overline{M}\right)} \overline{X}_{C(M)}$ it follows that

$$\mathfrak{Rep}_{\overline{H}}(\{\xi_n\}) \in 1_{\widetilde{\Omega}_{\widetilde{x}}} \overline{H} =_{C(M)} X_{C_0(\widetilde{M})} \otimes_{C\left(\overline{M}\right)} \overline{H} = \widetilde{H}.$$

$\square$

Remark 6.23. The sequence is local because the operator $\widetilde{\not{D}}$ can be defined as gluing of local operators, i.e. operators defined on one-to-one subsets.

6.24. Let M be a manifold with a Spin$^{\mathbf{c}}$ structure and $m \in \mathbb{N}$ is such that $\dim\ M = 2m$ or $\dim\ M = 2m+1$. If (M, π, S) is the spinor bundle and $x \in M$ is any point then from [18] it follows that $S_x = \pi^{-1}(x)$ is a complex Hilbert space of dimension 2^m and there is a natural representation $\rho_x : C^\infty(M) \to B(S_x)$ and

$$\|a\| = \sup_{x \in M} \|\rho_x(a)\|.$$

For any $x \in M$ there is a representation $\rho^1_x : C^\infty(M) \to B\left(S^2_x\right)$ which corresponds to the representation $\pi^1 : C^\infty(M) \to B\left(\left(L^2(M, S)\right)^2\right)$ given by

$$\pi^1(a) = \begin{pmatrix} a & 0 \\ [\not{D}, a] & a \end{pmatrix} \in B\left(\left(L^2(M, S)\right)^2\right).$$

Similarly to 5.1 for any $s \in \mathbb{N}$ and $x \in M$ we can inductively define $\rho^s_x : C^\infty(M) \to B\left(S^{2^s}_x\right)$ which corresponds to $\pi^s : C^\infty(M) \to B\left(\left(L^2(M, S)\right)^{2^s}\right)$ given by

$$\pi^s(a) = \begin{pmatrix} \pi^{s-1}(a) & 0 \\ \left[\not{D}, \pi^{s-1}(a)\right] & \pi^{s-1}(a) \end{pmatrix} \qquad (80)$$

Definition 6.25. Let M be an orientable Riemannian manifold with a Spin$^{\mathbf{c}}$ structure and let $\widetilde{\pi} : \widetilde{M} \to M$ be a covering projection. There is the unbounded Dirac operator $\not{D}$ on $L^2(M, S)$ and let $\widetilde{\not{D}}$ be the $\widetilde{\pi}$-pullback of $\not{D}$. An element $\xi \in \mathscr{L}^2\left(\widetilde{M}_M\right)$ of associated with $\widetilde{\pi} : \widetilde{M} \to M$ Hilbert $C(M)$-module is said to be *smooth* if following conditions hold

1. $\xi \in C^{\infty}\left(\widetilde{M}\right)$.
2. If $x \in M$ then for any $s \in \mathbb{N}$ and following condition hold

$$\varphi_{\xi}\left(x\right) = \sum_{y \in \widetilde{\pi}^{-1}(x)} \left\|\rho^{s}_{y}\left(\xi\right)\right\|^{2};\ \forall x \in M$$

then $\varphi_{\xi} \in C\left(M\right)$.

Definition 6.26. If $\Xi \subset \mathscr{L}^{2}\left(\widetilde{M}_{M}\right)$ be a linear span of smooth elements then there is the Fréchet topology on Ξ induced by seminorms $\left\|\cdot\right\|_{s}$ given by

$$\left\|\xi\right\|_{s} = \sup_{x} \sqrt{\sum_{y \in \widetilde{\pi}^{-1}(x)} \left\|\rho^{s}_{y}\left(\xi\right)\right\|^{2}}.$$

The completion of Ξ with respect to this the Fréchet topology is said to be the *smooth module associated* with $\widetilde{\pi} : \widetilde{M} \to M$. Denote by $\mathscr{L}^{2}_{\infty}\left(\widetilde{M}_{M}\right)$ the smooth module associated with $\widetilde{\pi} : \widetilde{M} \to M$.

6.27. From [36] it follows that Reimannian metric satisfies to the following condition

$$\mathfrak{dist}\left(x, y\right) = \sup\left\{\left|a\left(x\right) - a\left(y\right)\right| \ \middle|\ a \in C^{\infty}\left(M\right),\ \left\|\left[\not{D}, a\right]\right\| \le 1\right\}.$$

Conversely if $\left\|\left[\not{D}, a\right]\right\| = C$ then

$$\left|a\left(x\right) - a\left(y\right)\right| \le C\ \mathfrak{dist}\left(x, y\right).$$

Similarly if $\pi^{s}(a)$ is given by (80) and $\left\|\left[\not{D}, \pi^{s+1}(a)\right]\right\| = C_{s+1}$ then for any $x, y \in M$

$$\left\|\pi^{s}(a)\left(x\right) - \pi^{s}(a)\left(y\right)\right\| \le C_{s+1}\ \mathfrak{dist}\left(x, y\right). \quad (81)$$

Lemma 6.28. *Any element $\xi \in \mathscr{L}^{2}_{\infty}\left(\widetilde{M}_{M}\right)$ corresponds to the function $a \in C_{0}\left(\widetilde{M}\right)$ such that for any $s \in \mathbb{N}$ the function $f^{s} : \widetilde{M} \to \mathbb{R}$ given by*

$$f^{s}\left(x\right) = \left\|\rho^{s}_{x}\left(a\right)\right\|^{2}$$

is continuous.

Proof. From the Lemma 6.4 it follows that $\mathscr{L}^{2}\left(\widetilde{M}_{M}\right) \subset C_{0}\left(\widetilde{M}\right)$, so any element $\xi \in \mathscr{L}^{2}_{\infty}\left(\widetilde{M}_{M}\right)$ corresponds to the function $a \in C_{0}\left(\widetilde{M}\right)$. Now this lemma follows from the inequality (81). □

Definition 6.29. Let us consider the situation of the Definition 6.25. We say that the element $a \in C^{\infty}\left(\widetilde{M}\right)$ is *zero at infinity with derivations* if for any $s \in \mathbb{N}$, $\varepsilon > 0$ there is a compact set $K_{s,\varepsilon} \subset \widetilde{M}$ such that

$$\left\|\rho^{s}_{x}\left(a\right)\right\| < \varepsilon;\ \forall x \in \widetilde{M} \backslash K_{s,\varepsilon}.$$

Algebra of zero an infinity with derivations elements will be denoted by $C^{\infty}_{0}\left(\widetilde{M}_{M}\right)$.

Lemma 6.30. *Let M be an orientable Riemannian manifold with a* $\mathrm{Spin}^{\mathbf{c}}$ *structure. If $\widetilde{\pi}: \widetilde{M} \to M$ is a covering projection and $G\left(\widetilde{M} \mid M\right)$ is countable then $\mathscr{L}^2_{\infty}\left(\widetilde{M}_M\right) \subset C_0^{\infty}\left(\widetilde{M}_M\right)$.*

Proof. Proof of this lemma is similar to the proof of the Lemma 6.4. Let $\left\{\widetilde{U}_{\iota} \subset \widetilde{M}\right\}_{\iota \in I}$ be a basis of the fundamental covering of $\widetilde{\pi}: \widetilde{M} \to M$. Since M is compact we can select finite family $\left\{\widetilde{U}_{\iota} \subset \widetilde{M}\right\}_{\iota \in I}$, i.e. $\left\{\widetilde{U}_{\iota}\right\}_{\iota \in I} = \left\{\widetilde{U}_1, ..., \widetilde{U}_n\right\}$. Let

$$G_1 \leftarrow G_2 \leftarrow ...$$

be a coherent sequence of finite groups $G_i = G\left(M_i | M\right)$ with epimorphisms $h_i : G \to G_i$, and let $\left\{G^k \subset G\right\}_{k \in \mathbb{N}}$ be a G-covering (See the Definition 1.2). If $\widetilde{V} = \bigcup_{i=1,...,n} \widetilde{U}_i$ and $K = \mathfrak{cl}\left(\widetilde{V}\right)$ is the closure of $\widetilde{V}$ then K is compact. For any $k \in \mathbb{N}$ the set $K_k = G^k K$ is a finite union of compact sets, whence K_k is compact. If $\widetilde{V}_k = \bigcup_{k \in \mathbb{N}} G^k \widetilde{V}$ then from the Definition 3.6 it follows that $\widetilde{M} = \bigcup_{k \in \mathbb{N}} \widetilde{V}_k$. Let $\varphi \in \mathscr{L}^2_{\infty}\left(\widetilde{M}_M\right)$ be such that $\varphi \notin C_0^{\infty}\left(\widetilde{M}_M\right)$. From the Definition 6.29 it follows that there are $s \in \mathbb{N}$ and $\varepsilon > 0$ such that for any compact set $K \subset \widetilde{U}$ there is $\widetilde{x} \in \widetilde{\mathcal{X}} \backslash K$ such that $\left\|\rho^s_{\widetilde{x}}\left(\varphi\right)\right\| > \varepsilon$, where $\rho^s_{\widetilde{x}}$ is defined in 6.24. From 6.27 it follows that for any $s \in \mathbb{N}$ the function $\widetilde{x} \mapsto \left\|\rho^s_{\widetilde{x}}\right\|$ is continuous. Let us define a sequence $\left\{\widetilde{x}_i \in \widetilde{M}\right\}_{i \in \mathbb{N}}$ such that $\left|\rho^s_x\left(\widetilde{x}\right)\right| > \varepsilon$ and $\widetilde{x}_i \in \widetilde{M} \backslash K_i$. There is a sequence $\left\{x_i \in M\right\}_{i \in \mathbb{N}}$ given by $x_i = \pi\left(\widetilde{x}_{i_j}\right)$. Since M is compact the sequence $\left\{x_i\right\}_{i \in \mathbb{N}}$ contains a convergent subsequence $\left\{x_{i_j}\right\}_{j \in \mathbb{N}}$. Let $x = \lim_{j \to \infty} x_{i_j}$. Let $\widetilde{x} \in \widetilde{M}$ be such that $\widetilde{\pi}\left(\widetilde{x}\right) = x$ and $\widetilde{x} \in \widetilde{V}$. From the Definitions 6.25, 6.26 it follows that the series

$$\sum_{g \in G} \left\|\rho^s_{g\widetilde{x}}\left(\varphi\right)\right\|^2$$

is convergent, whence there is $r \in \mathbb{N}$ such that

$$\sum_{g \in G \backslash G^r} \left\|\rho^s_{g\widetilde{x}}\left(\varphi\right)\right\|^2 < \frac{\varepsilon^2}{2}. \tag{82}$$

If $\widetilde{W}$ is an open connected neighborhood of $\widetilde{x}$ which is mapped homeomorphicaly onto $W = \widetilde{\pi}\left(\widetilde{W}\right)$ and $\widetilde{W} \subset \widetilde{V}$ then there is a real continuous function $\psi : \widetilde{W} \to \mathbb{R}$ given by

$$\psi\left(y\right) = \sum_{g \in G \backslash G^r} \left\|\rho^s_{g\widetilde{y}}\left(\varphi\right)\right\|^2; \text{ where } \widetilde{y} \in \widetilde{W} \text{ and } \widetilde{\pi}\left(\widetilde{y}\right) = y.$$

There is $r \in \mathbb{N}$ such that $i_r > r$ and $x_{i_j} \in W$ for any $j \geq r$. If $j > r$ then from $\left|\varphi\left(\widetilde{x}_{i_j}\right)\right| > \varepsilon$ and $\widetilde{x}_{i_j} \notin K_r$ it follows that

$$\psi\left(x_{i_j}\right) = \sum_{g \in G \backslash G^r} \left\|\rho^s_{g\widetilde{x}'_{i_j}}\left(\varphi\right)\right\|^2 \geq \left\|\rho^s_{g\widetilde{x}_{i_j}}\left(\varphi\right)\right\|^2 > \varepsilon^2; \text{ where } \widetilde{x}'_{i_j} \in \widetilde{W} \text{ and } \widetilde{\pi}\left(\widetilde{x}'_{i_j}\right) = x_{i_j}.$$

Since ψ is continuous and $x = \lim_{j\to\infty} x_{i_j}$ we have $\psi(x) > \varepsilon^2$. This fact contradicts to the equation (82), and the contradiction proves the lemma.

$\square$

Lemma 6.31. *If $X^\infty_{C^\infty(M)}$ is the connected smooth module (Definition 5.29) of the coherent sequence described in the Lemma 6.21 then there is the natural isomorphism of $C^\infty(M)$-modules*

$$\mathscr{L}^2_\infty\left(\widetilde{M}_M\right) \approx X^\infty_{C^\infty(M)}.$$

Proof. 1) Inclusion $\mathscr{L}^2_\infty\left(\widetilde{M}_M\right) \subset X^\infty_{C^\infty(M)}$.
Let $G = G\left(\widetilde{M}|M\right)$ be the group of covering transformations. Since M is compact there is a finite basis $\left\{\widetilde{U}_1, ..., \widetilde{U}_n\right\}$ of the fundamental covering of $\widetilde{\pi} : \widetilde{M} \to M$. From the Proposition 1.38 it follows that there exist a partition of unity

$$1_{C_b(\widetilde{M})} = \sum_{g\in G}\sum_{i=1}^n g\widetilde{a}_i = \sum_{(g,\iota)\in G\times\{1,...,n\}} \widetilde{a}_{(g,\iota)}$$

dominated by $\left\{\widetilde{U}_1, ..., \widetilde{U}_n\right\}$ such that $\widetilde{a}_i \in C^\infty\left(\widetilde{M}\right)$ for any $i \in \{1, ..., n\}$.
If $\widetilde{e}_i = \sqrt{\widetilde{a}_i}$ then $\widetilde{e}_i \in C^\infty\left(\widetilde{M}\right)$ and from the Corollary 4.30 it follows that

$$\sum_{i\in\{1,...,n\},g\in G} g\mathfrak{Rep}\left(\widetilde{e}_i\right)\rangle\langle g\mathfrak{Rep}\left(\widetilde{e}_i\right) = 1_{M\left(\mathcal{K}\left(X_{C(M)}\right)\right)}. \tag{83}$$

From $\widetilde{e}_i \in C^\infty\left(\widetilde{M}\right)$ it follows that the descent $\mathfrak{Desc}\left(\widetilde{e}_i\right)$ of $\widetilde{e}_i$ (Definition 6.7) is a smooth coherent sequence (Definition 5.27), whence $\mathfrak{Rep}\left(\widetilde{e}_i\right) \in X^\infty_{C^\infty(M)}$. If for any $a \in C^\infty(M)$ and $s \in \mathbb{N}$ we denote

$$\|a\|_s = \|\pi^s(a)\|; \text{ where } \pi^s \text{ is given by (80)}$$

then from

$$C_s = \max_{i\in\{1,...,n\}} \|\widetilde{e}_i\|_s.$$

it follows that

$$\|\widetilde{e}_i a\|_s < C_s \|a\|_s,$$

whence $\mathfrak{Rep}\left(g\widetilde{e}_i\right) a \in X^\infty_{C^\infty(M)}$. If $\varphi \in \mathscr{L}^2_\infty\left(\widetilde{M}_M\right)$ then form $\mathscr{L}^2_\infty\left(\widetilde{M}_M\right) \subset \mathscr{L}^2\left(\widetilde{M}_M\right)$ and from the Lemma 6.4 it follows that φ defines a unique element $\xi_\varphi \in X_{C(M)}$. From (83) it follows that

$$\xi_\varphi = \sum_{i\in\{1,...,n\},g\in G} g\mathfrak{Rep}\left(\widetilde{e}_i\right)\left\langle \xi, g\mathfrak{Rep}\left(\widetilde{e}_i\right)\right\rangle_{X_{C(M)}}$$

All summands of the above series belong to $X^\infty_{C^\infty(M)}$. Let us prove that the series is convergent in the Fréchet topology given by the Definition 5.29. Let $s \in \mathbb{N}$ be any natural

number and $\varepsilon > 0$. From the Definition 6.25 it follows that there is a compact set K such that for any $\widetilde{x} \in \widetilde{M} \backslash K$ following condition hold

$$\rho^s_{\widetilde{x}}(\varphi) < \frac{\varepsilon}{C_s}$$

Since K is compact there is a finite subset $G' \in G$ such that

$$\left(\sum_{g \in G'} \sum_{i=1}^{n} g\widetilde{a}_i\right)(K) = \{1\}.$$

From above equations it follows that

$$\left\| \xi_\varphi - \sum_{g \in G'} \sum_{i=1}^{n} g\mathfrak{Rep}\left(\widetilde{e}_i\right) \left\langle \xi, g\mathfrak{Rep}\left(\widetilde{e}_i\right)\right\rangle_{\overline{X}_A} \right\|_s < \varepsilon. \tag{84}$$

So the series is convergent in the Fréchet topology, whence $\varphi_\xi \in \overline{X}^\infty_{C^\infty(M)}$.

2) Inclusion $X^\infty_{C^\infty(M)} \subset \mathscr{L}^2_\infty\left(\widetilde{M}_M\right)$.

If $\xi \in X^\infty_{C^\infty(M)}$ then form $X^\infty_{C^\infty(M)} \subset X_{C(M)}$ and from the Lemma 6.4 it follows that ξ defines a unique element $\varphi_\xi \in \mathscr{L}^2\left(\widetilde{M}_M\right)$. If $\varphi_\xi \notin \mathscr{L}^2_\infty\left(\widetilde{M}_M\right)$ then there are $s \in \mathbb{N}$ and $\varepsilon > 0$ such that for any compact set K there is $\widetilde{x} \in \widetilde{M} \backslash K$ such that a following condition holds

$$\left\|\rho^s_{\widetilde{x}}\left(\widetilde{\varphi}_\xi\right)\right\|_s > \varepsilon.$$

Let us define a sequence $\left\{\widetilde{x}_i \in \widetilde{M}\right\}_{i \in \mathbb{N}}$ such that $\left\|\left|\rho^s_{\widetilde{x}_i}(\varphi)\right|\right\| > \varepsilon$ and $\widetilde{x}_i \in \widetilde{M} \backslash K_i$. There is a sequence $\{x_i \in \mathcal{X}\}_{i \in \mathbb{N}}$ given by $x_i = \pi\left(\widetilde{x}_{i_j}\right)$. Since M is compact the sequence $\{x_i\}_{i \in \mathbb{N}}$ contains a convergent subsequence $\left\{x_{i_j}\right\}_{j \in \mathbb{N}}$. Let $x = \lim_{j \to \infty} x_{i_j}$. Let $\widetilde{x} \in \widetilde{M}$ be such that $\widetilde{\pi}(\widetilde{x}) = x$ and $\widetilde{x} \in \widetilde{V}$. From (69) it follows that the series

$$\sum_{g \in G} \left|\varphi\left(g\widetilde{x}\right)\right|^2$$

is convergent, whence there is $r \in \mathbb{N}$ such that

$$\sum_{g \in G \backslash G^r} \left\|\rho^s_{g\widetilde{x}}(\varphi)\right\|^2 < \frac{\varepsilon^2}{2}. \tag{85}$$

Let $\widetilde{W}$ be an open connected neighborhood of $\widetilde{x}$ which is mapped homeomorphicaly onto $W = \widetilde{\pi}\left(\widetilde{W}\right)$ such that following conditions hold:

1. If $\widetilde{y} \in \widetilde{W}$ then $\mathfrak{dist}\left(\widetilde{y}, \widetilde{x}\right) < \frac{\varepsilon}{4\left\|\left[\not{D}\ , \pi^{s+1}(\varphi)\right]\right\|}$. From this condition and (81) it follows that
$$\left\|\rho^s_{g\widetilde{y}}(\varphi) - \rho^s_{g\widetilde{x}}(\varphi)\right\| < \frac{\varepsilon}{4};\ \forall y \in \widetilde{W}. \tag{86}$$

2. $\widetilde{W} \subset \widetilde{V}$.

There is a real continuous function $\psi : \widetilde{W} \to \mathbb{R}$ given by

$$\psi(y) = \sum_{g \in G \backslash G^r} \left\| \rho^s_{g\widetilde{y}}(\varphi) \right\|^2; \text{ where } \widetilde{y} \in \widetilde{W} \text{ and } \widetilde{\pi}(\widetilde{y}) = y.$$

There is $s \in \mathbb{N}$ such that $i_s > r$ and $x_{i_j} \in W$ for any $j \geq s$. If $j > s$ then from $\left\| \rho^s_{g\widetilde{x}_{i_j}} \right\| > \varepsilon$ and $\widetilde{x}_{i_j} \notin K_r$ it follows that

$$\psi\left(x_{i_j}\right) = \sum_{g \in G \backslash G^r} \left\| \rho^s_{g\widetilde{x}'_{i_j}}(\varphi) \right\|^2 \geq \left\| \rho^s_{g\widetilde{x}_{i_j}}(\varphi) \right\|^2 > \varepsilon^2; \text{ where } \widetilde{x}'_{i_j} \in \widetilde{\mathcal{W}} \text{ and } \widetilde{\pi}\left(\widetilde{x}'_{i_j}\right) = x_{i_j}.$$

Since ψ is continuous and $x = \lim_{j \to \infty} x_{i_j}$ we have $\psi(x) > \varepsilon^2$. This fact contradicts to equations (85),(86), and the contradiction proves the lemma. $\square$

Lemma 6.32. *Let us consider the coherent sequence described in the Lemma 6.21. If $\mathcal{K}^\infty\left(X^\infty_{C^\infty(M)}\right)$ is the smoothly compact subalgebra (Definition 5.32) then*

$$C^\infty_0\left(\widetilde{M}_M\right) = C_0\left(\widetilde{M}\right) \bigcap \mathcal{K}^\infty\left(X_{C^\infty(M)}\right). \tag{87}$$

Proof. 1) Inclusion $C_0\left(\widetilde{M}\right) \bigcap \mathcal{K}^\infty\left(X_{C^\infty(M)}\right) \subset C^\infty_0\left(\widetilde{M}_M\right)$.
If $a \in C_0\left(\widetilde{M}\right) \bigcap \mathcal{K}^\infty\left(X_{C^\infty(M)}\right)$ given by

$$a = \sum_{i=1}^\infty \xi_i\rangle\langle\eta_i; \quad \xi_i, \eta_i \in \overline{X}^\infty_{C^\infty(M)}$$

then from (76) it follows that

$$a = \sum_{i=1}^\infty \xi^*_i \eta_i \tag{88}$$

where ξ_i, η_i are being regarded as elements of $C_0\left(\widetilde{M}\right)$. From the Lemma 6.30 it follows that $\xi_i\eta_i \in C^\infty_0\left(\widetilde{M}_M\right)$ whence $a_n = \sum_{i=1}^n \xi^*_i \eta_i \in C^\infty_0\left(\widetilde{M}_M\right)$ for any $n \in \mathbb{N}$. From the Definition 5.32 it follows that the series (88) is convergent in the Fréchet topology induced by seminorms $\|\cdot\|_s$, therefore $a \in C^\infty_0\left(\widetilde{M}_M\right)$.
2) Inclusion $C^\infty_0\left(\widetilde{M}_M\right) \subset C_0\left(\widetilde{M}\right) \bigcap \mathcal{K}^\infty\left(X_{C^\infty(M)}\right)$.
Let $a \in C^\infty_0\left(\widetilde{M}_M\right)$. If $\widetilde{e}_1, ..., \widetilde{e}_n \in C^\infty\left(\widetilde{M}\right)$ are defined in the Lemma 6.31 then following condition hold

$$a = \sum_{g \in G} \sum_{i=1}^n g\widetilde{e}_i \langle g\widetilde{e}_i, a\rangle_{\overline{X}_{C(M)}} = \sum_{g \in G} \sum_{i=1}^n g\widetilde{e}_i\rangle\langle g\widetilde{e}_i a = \sum_{g \in G} \sum_{i=1}^n \left(g\widetilde{e}_i\right)\left(g\widetilde{e}_i a\right)$$

and the series is convergent in the Fréchet topology induced by seminorms $\|\cdot\|_s$. From this fact and it follows that $a \in C_0\left(\widetilde{M}\right) \bigcap \mathcal{K}^\infty\left(X_{C^\infty(M)}\right)$.

$\square$

Lemma 6.33. *The coherent sequence described in the Lemma 6.21 is regular.*

Proof. For any $\widetilde{a} \in C_0(M)$ and $\varepsilon > 0$ there is a finite subset $G' \subset G$ such that

$$\left\|\widetilde{a} - \sum_{i=1}^{m} \sum_{g \in G'} a_{ig} \left(g\widetilde{e}_i\right)^2\right\| < \frac{\varepsilon}{2}$$

where $a_{ig} \in A$ and $\widetilde{e}_1, ..., \widetilde{e}_n \in C^\infty\left(\widetilde{M}\right)$ are defined in the Lemma 6.31. Since $C^\infty(M)$ is dense in $C(M)$ there are $a'_{ig} \in C^\infty(M)$ such that $\left\|a'_{ig} - a_{ig}\right\| \le \frac{\varepsilon}{2|G'|}$, so

$$\widetilde{a}' = \sum_{i=1}^{m} \sum_{g \in G''} a'_{ig} \left(g\widetilde{e}_i\right)^2 \in C_0^\infty\left(\widetilde{M}_M\right)$$

and

$$\left\|\widetilde{a}' - \sum_{i=1}^{m} \sum_{g \in G''} a_{ig} \left(g\widetilde{e}_i\right)\right\| < \frac{\varepsilon}{2}$$

In result we have $\|\widetilde{a} - \widetilde{a}'\| < \varepsilon$, whence $C^\infty\left(\widetilde{M}_M\right)$ is dense in $C\left(\widetilde{M}\right)$ because $a' \in C^\infty\left(\widetilde{M}_M\right)$.

$\square$

7 Covering projection of the noncommutative torus

7.1 Covering projection of the C^*-algebra

7.1. Let A_θ be a noncommutative torus generated by unitaries $u, v \in U(A_\theta)$ and let

$$\mathbb{Z}_{n_1} \times \mathbb{Z}_{m_1} \leftarrow ... \leftarrow \mathbb{Z}_{n_k} \times \mathbb{Z}_{m_k} \leftarrow ... \qquad (89)$$

be an infinite sequence of finite groups. Let $\theta_0 = \theta$ and let us inductively define θ_k for any $k \in \mathbb{N}$ and *-homomorphism $A_{\theta_{k-1}} \to A_{\theta_k}$ such that

1. $$\theta_k = \begin{cases} \frac{\theta + 2\pi i_1}{m_1 n_1} & k = 1 \\ \frac{\theta_{k-1} + 2\pi i_k}{\frac{m_k}{m_{k-1}} \frac{n_k}{n_{k-1}}} & k > 1 \end{cases}$$

   where $i_1, ..., i_k, ... \in \mathbb{N}^0$ arbitrary nonnegative numbers.

2. A_{θ_k} is generated by two unitary elements u_k, v_k and the *-homomorphism $A_{\theta_{k-1}} \to A_{\theta_k}$ is given by

$$u_{k-1} \mapsto u_k^m;\ v_{k-1} \mapsto v_k^n$$

where

$$m \text{ (resp. } n) = \begin{cases} m_1 \text{ (resp. } n_1) & k = 1 \\ \frac{m_k}{m_{k-1}} \text{ (resp. } \frac{n_k}{n_{k-1}}) & k > 1 \end{cases}$$

In [17] it is shown that the sequence

$$A_\theta = A_{\theta_0} \to A_{\theta_1} \to ... \tag{90}$$

is a composable sequence of finite noncommutative covering projections. If e_i (resp. e_i^n) is defined by (41) (43) then any $i, j \in \{1, 2\}$ following sequences

$$\Lambda_{i,j} = \left\{ e_i^{m_k}\left(u_{m_k}\right) e_j^{n_k}\left(v_{n_k}\right) \right\}_{k \in \mathbb{N}^0};\ \Lambda'_{i,j} = \left\{ e_j^{n_k}\left(v_{n_k}\right) e_i^{m_k}\left(u_{m_k}\right) \right\}_{k \in \mathbb{N}^0}$$

are coherent. If $I = \{1, 2\} \times \{1, 2\}$ and $\Lambda_{\iota=(i,j)} = \Lambda_{i,j}$ (resp. $\Lambda'_{\iota=(i,j)} = \Lambda'_{i,j}$) $\forall \iota \in I$ then elements $\xi_\iota = \mathfrak{Rep}\left(\Lambda_\iota\right)$, $\xi'_\iota = \mathfrak{Rep}\left(\Lambda'_\iota\right)$ satisfy conditions of the Corollary 4.30. If $\Xi = \{\xi_\iota\}_{\iota \in I}$ then from the Corollary 4.31 it follows that the set $\overline{G}\Xi A_\theta$ is dense in $\overline{X}_{A_\theta}$. For any $(x, y) \in \mathbb{R} \times \mathbb{R}$ let $(x, y) \bullet \mathfrak{Rep}\left(\Lambda_{i,j}\right)$ be given by

$$(x, y) \bullet \mathfrak{Rep}\left(\Lambda_{i,j}\right) = \mathfrak{Rep}\left(\left\{ e^{m_k}\left(\exp\left(\frac{ix}{m_k} \right) u_{m_k} \right) e^{m_k}\left(\exp\left(\frac{iy}{n_k} \right) v_{n_k} \right) \right\}_{k \in \mathbb{N}^0} \right); \tag{91}$$

Since a linear span of $\overline{G}\Xi A_\theta$ is dense in $\overline{X}_{A_\theta}$ the action of $\mathbb{R} \times \mathbb{R}$ can be uniquely extended to the continuous action $(\mathbb{R} \times \mathbb{R}) \times \overline{X}_A \to \overline{X}_A$, $\xi \mapsto (x, y) \bullet \xi$. This action induces a natural action $(\mathbb{R} \times \mathbb{R}) \times \overline{A}_\theta \to \overline{A}_\theta$ on disconnected algebra. Since both $\mathbb{R} \times \mathbb{R}$ and a maximal irreducible subalgebra algebra $\widetilde{A}_\theta \subset \overline{A}_\theta$ are connected a following condition hold

$$(\mathbb{R} \times \mathbb{R}) \bullet \widetilde{A} = \widetilde{A}. \tag{92}$$

If we include $\mathbb{Z} \times \mathbb{Z} \subset \overline{G}$ then

$$(i, j)a = (2\pi i, 2\pi j) \bullet a;\ \forall a \in \overline{a}$$

and from (92) it follows that $(\mathbb{Z} \times \mathbb{Z})\, \widetilde{A} = \widetilde{A}$. Otherwise if $g \in (\mathbb{Z} \times \mathbb{Z})$, $g' \in \overline{G} \backslash (\mathbb{Z} \times \mathbb{Z})$ and $\eta, \zeta \in \Xi$ then $\langle g\eta, g'\zeta \rangle_{\overline{X}_{A_\theta}} = 0$. From this fact it follows that the covering transformation group G is equal to $G = \mathbb{Z} \times \mathbb{Z}$ and a linear span of $(\mathbb{Z} \times \mathbb{Z})\ \Xi\ A_\theta$ is dense in X_{A_θ}. There is a noncummutative infinite covering projection $\left(A_\theta,\ \widetilde{A}_\theta,\ {}_{\widetilde{A}_\theta}X_{A_\theta},\ \mathbb{Z} \times \mathbb{Z}\right)$ of the sequence (90).

Lemma 7.2. *If for $i, j \in \{1, 2\}$ elements $\xi_{ij}, \eta_{ij} \in_{\widetilde{A}_\theta} X_{A_\theta}$ are given by*

$$\xi_{ij} = \mathfrak{Rep}\left(\left\{ e_i^{m_k}\left(u_{m_k}\right) e_j^{n_k}\left(v_{n_k}\right) \right\}_{k \in \mathbb{N}^0} \right),\ \eta_{ij} = \left\{ e_i^{m_k}\left(u^*_{m_k}\right) e_j^{n_k}\left(v^*_{n_k}\right) \right\}_{k \in \mathbb{N}^0} \in_{\widetilde{A}_\theta} X_{A_\theta}$$

then $\xi_{ij}\rangle\langle\eta_{ij} \in\ \overline{A}_\theta$.

Proof. From the Definition 4.27 if follows that we need to show that

$$\xi_{ij}\rangle\langle\eta_{ij} \in \left(\bigcup_{k\in\mathbb{N}} A_{\theta_k}\right)''.$$

If $a_k \in A_{\theta_k}$ is given by

$$a_k = e_i^{m_k}\left(u_{m_k}\right) e_j^{n_k}\left(v_{n_k}\right) e_i^{n_k}\left(v_{n_k}\right) e_j^{m_k}\left(u_{m_k}\right)$$

then it is clear that $a_k = \bigcup_{k\in\mathbb{N}} A_{\theta_k}$ and

$$\lim_{k\to\infty} a_k = \xi_{ij}\rangle\langle\eta_{ij}$$

where we mean convergence in the weak topology. □

Corollary 7.3. *The sequence* (90) *of covering projections is faithful.*

Proof. The natural homomorphisms $\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}_{m_k}\times\mathbb{Z}_{n_k}$ are epimorphic. From (49) it follows that

$$\sum_{g\in\mathbb{Z}_{m_k}\times\mathbb{Z}_{n_k};\ ,i,j\in\{1,2\}} \left(g e_i^{m_k}\left(u_{m_k}\right) e_j^{n_k}\left(v_{n_k}\right)\right)\left(g e_i^{n_k}\left(v_{n_k}\right) e_j^{m_k}\left(u_{m_k}\right)\right) = 1_{A_{\theta_k}},$$

whence for any nonzero $a \in A_{\theta_k}$ there are $g_0 \in \mathbb{Z}_{m_k}\times\mathbb{Z}_{n_k}$ and i_0, j_0 such that $g_0 e_{i_0}^{n_k}\left(v_{n_k}\right) e_{j_0}^{m_k}\left(u_{m_k}\right) a \neq 0$. If $\widetilde{g} \in \mathbb{Z}\times\mathbb{Z}$ is such that $h_k\left(\widetilde{g}\right) = g_0$ then $g_0\xi_{i_0j_0}\rangle\langle g_0\eta_{i_0j_0} a \neq 0$. But from the Lemma 7.2 it follows that $g_0\xi_{i_0j_0}\rangle\langle g_0\eta_{i_0j_0} \in \overline{A}_\theta$. So the right action of A_{θ_k} on $\overline{A}_\theta$ is faithful. Similarly we can prove that the right action is also faithful. □

7.4. From the Corollary 4.30 it follows that

$$1_{M\left(\widetilde{A}_\theta\right)} = \sum_{g\in\mathbb{Z}\times\mathbb{Z}}\ \sum_{i,j\in\{1,2\}} g\xi_{ij}\rangle\langle g\eta_{ij}$$

in sense of the strict convergence. Any $\kappa \in \mathcal{K}\left({}_{\widetilde{A}_\theta}X_{A_\theta}\right)$ can be represented as sum of the norm convergent series

$$\kappa = \sum_{k\in\mathbb{N}} \phi_k\rangle\langle\psi_k$$

whence

$$\kappa = \left(\sum_{g\in\mathbb{Z}\times\mathbb{Z}}\ \sum_{i,j\in\{1,2\}} g\xi_{ij}\rangle\langle g\eta_{ij}\right)\sum_k \phi_k\rangle\langle\psi_k \left(\sum_{g\in\mathbb{Z}\times\mathbb{Z}}\ \sum_{i,j\in\{1,2\}} g\xi_{ij}\rangle\langle g\eta_{ij}\right) =$$

$$= \sum_{g,g'\in\mathbb{Z}\times\mathbb{Z}}\ \sum_{i,j,i',j'\in\{1,2\}} g\xi_{ij}\rangle a_{ijgi'j'g'}\langle g'\eta_{i'j'} = \sum_{g,g'\in\mathbb{Z}\times\mathbb{Z}}\ \sum_{i,j,i',j'\in\{1,2\}} g\xi_{ij}a_{ijgi'j'g'}\rangle\langle g'\eta_{i'j'}$$

where

$$a_{ijgi'j'g'} = \sum_{k\in\mathbb{N}} \left\langle g\eta_{ij} \ , \ \kappa g'\xi_{i'j'} \right\rangle_{\widetilde{A}_\theta X_{A_\theta}} \in A_\theta.$$

Let $\left\{G^k \subset \mathbb{Z}\times\mathbb{Z}\right\}_{k\in\mathbb{N}}$ be $\mathbb{Z}\times\mathbb{Z}$ covering of the sequence (89) (Definition 1.2). Since κ is compact for any $\varepsilon > 0$ there is $l \in \mathbb{N}$ such that if

$$\overline{\kappa} = \sum_{g,g'\in G^l} \ \sum_{i,j,i',j'\in\{1,2\}} g\xi_{ij}\rangle a_{ijgi'j'g'} \langle g'\eta_{i'j'}$$

then

$$\|\kappa - \overline{\kappa}\| < \frac{\varepsilon}{2}.$$

If $\kappa \in \widetilde{A}_\theta$ then there is a sequence $\left\{a_k \in A_{\theta_k}\right\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} a_k = \kappa$$

in the weak topology. For any $i, j, i', j' \in \{1,2\}$ and $g, g' \in G^l$ we will define the element $a^k_{ijgi'j'g'} \in A_\theta$ given by

$$a^k_{ijgi'j'g'} = \left\langle h_k\left(g\right)\left(e^{n_k}_j\left(v_{n_k}\right)e^{m_k}_i\left(u_{m_k}\right)\right) \ , \ a_k\left(h_k\left(g'\right)\left(e^{m_k}_{i'}\left(u_{m_k}\right)e^{n_k}_{j'}\left(v_{n_k}\right)\right)\right)\right\rangle_{A_{\theta_k}}$$

where $h_k : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}_{m_k}\times\mathbb{Z}_{n_k}$ is the natural epimorphism of groups. From $\lim_{k\to\infty} a_k = \kappa$ it follows that

$$\lim_{k\to\infty} a^k_{ijgi'j'g'} = a_{ijgi'j'g'}$$

and there is $k_0 \in \mathbb{N}$ such that

$$\left\|a_{ijgi'j'g'} - a^{k_0}_{ijgi'j'g'}\right\| < \frac{\varepsilon}{16\left|G^l\right|}$$

and if

$$\kappa_{\text{fin}} = \sum_{g,g'\in G^l} \ \sum_{i,j,i',j'\in\{1,2\}} g\xi_{ij}\rangle a^{k_0}_{ijgi'j'g'} \langle g'\eta_{i'j'}$$

then

$$\|\kappa - \kappa_{\text{fin}}\| < \varepsilon. \tag{93}$$

If $\{b_r \in A_{\theta_r}\}_{r\in\mathbb{N}}$ is the sequence given by $b_r = 0$ for $r \le k$ and

$$b_r = \sum_{g,g'\in G^k} \ \sum_{i,j,i',j'\in\{1,2\}} h_r\left(g\right)\left(e^{n_r}_j\left(v_{n_r}\right)e^{m_r}_i\left(u_{m_r}\right)\right) a^{k_0}_{ijgi'j'g'} \left(h_r\left(g'\right)\left(e^{m_r}_{i'}\left(u_{m_r}\right)e^{n_r}_{j'}\left(v_{n_r}\right)\right)\right)$$

for $r > k$ then $\lim_{r\to\infty} b_r = \kappa_{\text{fin}}$ in the weak topology i.e. $\kappa_{\text{fin}} \in \widetilde{A}_\theta$.

Lemma 7.5. *The space of operators κ_{fin} given by construction* 7.4 *is dense in $\widetilde{A}_\theta$,*

Proof. Follows from the Equation (93). $\square$

7.2 Covering projection of the spectral triple

7.6. *Noncommutative torus as a spectral triple* [18,36].
There is a state $\tau_0 : A_\theta \to \mathbb{C}$ given by

$$\tau_0 \left(\sum_{r,s} a_{rs} u^r v^s \right) = a_{00}.$$

.
The GNS representation space $L^2(A_\theta, \tau_0)$ may be described as the completion of the vector space A_θ with respect to the Hilbert norm

$$\|a\|_2 = \tau_0 \left(\sqrt{a^* a} \right).$$

Denote by $\underline{a}$ the image in $L^2(A_\theta, \tau_0)$ of $a \in A_\theta$. The Hilbert product on $L^2(A_\theta, \tau_0)$ is given by

$$(\underline{a}, \underline{b}) = \tau_0 (a^* b).$$

Since $\underline{1}$ is cyclic and separating the Tomita involution is given by

$$J_0(\underline{a}) = \underline{a^*}.$$

To define structure of spectral triple we shall introduce double GNS Hilbert space $H = L^2(A_\theta, \tau_0) \oplus L^2(A_\theta, \tau_0)$ and define

$$J = \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix}$$

There are two derivatives δ_1, δ_2 given by

$$\delta_1 \left(\sum_{r,s} a_{r,s} u^r v^s \right) = \sum_{rs} 2\pi i r a_{rs} u^r v^s,$$

$$\delta_2 \left(\sum_{r,s} a_{r,s} u^r v^s \right) = \sum_{rs} 2\pi i s a_{rs} u^r v^s.$$

which satisfy Leibniz rule, i.e.

$$\delta_j(ab) = (\delta_j a)b = a(\delta_j b);\ (j = 1, 2;\ a, b \in \mathcal{A}_\theta).$$

There are derivations

$$\partial = \partial_\tau = \delta_1 + \tau\delta_2;\ (\tau \in \mathbb{C},\ \mathrm{Im}(\tau) \neq 0), \tag{94}$$

$$\partial^\dagger = -\delta_1 - \overline{\tau}\delta_2.$$

Actions of $\mathcal{A}_\theta$ and Dirac operator D on H are given by

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & \underline{\partial}^\dagger \\ \underline{\partial} & 0 \end{pmatrix}.$$

There is a subalgebra $\mathcal{A}_\theta \subset A_\theta$ given by

$$\mathcal{A}_\theta = \left\{ a \in A_\theta \mid a = \sum_{r,s} a_{rs} u^r v^s \text{ \& } \sup_{r,s\in\mathbb{Z}} \left(1 + r^2 + s^2\right)^k |a_{rs}| < \infty, \ \forall k \in \mathbb{N} \right\}.$$

and a spectral triple $(\mathcal{A}_\theta, H, D)$.

7.7. *Spectral triple of a noncommutative covering projection.* Let $m, n, k \in \mathbb{N}$, $\widetilde{\theta} = \frac{\theta + 2\pi k}{mn}$ and let $A_\theta \to A_{\widetilde{\theta}}$ be a *-homomorphism given by

$$u \mapsto \widetilde{u}^m; \ v \mapsto \widetilde{v}^n$$

where $\widetilde{u}, \widetilde{v} \in U\left(A_{\widetilde{\theta}}\right)$ are unitary generators of $A_{\widetilde{\theta}}$. Similarly to 7.6 we can define a smooth algebra

$$\mathcal{A}_{\widetilde{\theta}} = \left\{ a \in A_{\widetilde{\theta}} \mid a = \sum_{r,s} a_{rs} \widetilde{u}^r \widetilde{v}^s \text{ \& } \sup_{r,s\in\mathbb{Z}} \left(1 + r^2 + s^2\right)^k |a_{rs}| < \infty, \ \forall k \in \mathbb{N} \right\}.$$

and a state

$$\widetilde{\tau}_0 \left(\sum_{r,s} a_{rs} \widetilde{u}^r \widetilde{v}^s \right) = a_{00}.$$

The GNS representation space $L^2(A_{\widetilde{\theta}}, \widetilde{\tau}_0)$ may be described as the completion of the vector space $A_{\widetilde{\theta}}$ in the Hilbert norm

$$\|a\|_2 = \widetilde{\tau}\left(\sqrt{a^* a}\right).$$

Denote by $\underline{a}$ the image in $L^2\left(A_{\widetilde{\theta}}, \widetilde{\tau}_0\right)$ of $a \in A_{\widetilde{\theta}}$. Similarly to 7.6 we define

$$\widetilde{J}_0(\underline{a}) = \underline{a^*}.$$

To define structure of spectral triple we shall introduce double GNS Hilbert space $\widetilde{H} = L^2\left(A_{\widetilde{\theta}}, \widetilde{\tau}_0\right) \oplus L^2\left(A_{\widetilde{\theta}}, \widetilde{\tau}_0\right)$ and define

$$\widetilde{J} = \begin{pmatrix} 0 & -\widetilde{J}_0 \\ \widetilde{J}_0 & 0 \end{pmatrix}$$

Thee are two derivatives $\widetilde{\delta}_1$, $\widetilde{\delta}_2$

$$\widetilde{\delta}_1 \left(\sum_{r,s} a_{r,s} \widetilde{u}^r \widetilde{v}^s \right) = \frac{1}{m} \sum_{rs} 2\pi i r a_{rs} \widetilde{u}^r \widetilde{v}^s,$$

$$\widetilde{\delta}_2\left(\sum_{r,s} a_{r,s}\widetilde{u}^r\widetilde{v}^s\right) = \frac{1}{n}\sum_{rs} 2\pi i s a_{rs}\widetilde{u}^r\widetilde{v}^s,$$

which satisfy Leibniz rule. There are derivations

$$\widetilde{\partial} = \widetilde{\partial}_\tau = \widetilde{\delta}_1 + \tau\widetilde{\delta}_2,$$

$$\widetilde{\partial}^\dagger = -\widetilde{\delta}_1 - \overline{\tau}\widetilde{\delta}_2.$$

Hilbert space of the spectral triple $\widetilde{H}$, action of $\mathcal{A}_{\widetilde{\theta}}$ and Dirac operator $\widetilde{D}$ are given by

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

$$\widetilde{D} = \begin{pmatrix} 0 & \underline{\widetilde{\partial}}^\dagger \\ \underline{\widetilde{\partial}} & 0 \end{pmatrix}.$$

In result we have a spectral triple $\left(\mathcal{A}_{\widetilde{\theta}}, \widetilde{H}, \widetilde{D}\right)$. An application of this construction gives a sequence of spectral triples $\{(\mathcal{A}_{\theta_n}, H_n, D_n)\}_{n\in\mathbb{N}^0}$ such that A_{θ_n} is the C^*-completion of $\mathcal{A}_{\theta_n}$ and A_{θ_n} is a member of the sequence (90).

Lemma 7.8. *In the situation described in 7.7 the sequence* $\{(\mathcal{A}_{\theta_n}, H_n, D_n)\}_{n\in\mathbb{N}^0}$ *is coherent.*

Proof. We need to prove conditions 1-6 of the Definition 5.2. Conditions 1-3 follow from the construction 7.7. Let $\left(A_\theta, \widetilde{A}_\theta, \mathbb{Z}_m \times \mathbb{Z}_n\right)$ be a finite noncommutative covering projection from 7.7. If $g = (\overline{p}, \overline{q}) \in \mathbb{Z}_m \times \mathbb{Z}_n$ then

$$g\left(\sum_{r,s} a_{rs}\widetilde{u}^r\widetilde{v}^s\right) = \sum_{r,s} e^{\frac{2\pi i p r}{m}} e^{\frac{2\pi i q s}{n}} a_{rs}\widetilde{u}^r\widetilde{v}^s.$$

From $|a_{rs}| = \left|a_{rs}e^{\frac{2\pi i p r}{m}} e^{\frac{2\pi i q s}{n}}\right|$ it follows that

$$\sup_{r,s\in\mathbb{Z}}\left(1 + r^2 + s^2\right)^k |a_{rs}| < \infty;\ \forall k \in \mathbb{N} \Rightarrow$$
$$\Rightarrow \sup_{r,s\in\mathbb{Z}}\left(1 + r^2 + s^2\right)^k \left|e^{\frac{2\pi i p r}{m}} e^{\frac{2\pi i q s}{n}} a_{rs}\right| < \infty;\ \forall k \in \mathbb{N},$$

whence $g\mathcal{A}_{\widetilde{\theta}} = \mathcal{A}_{\widetilde{\theta}}$ and the Condition 4 of 7.7 is proven. From $\mathcal{H}^\infty = \bigcap_{k\in\mathbb{N}} \operatorname{Dom} D^k = \mathcal{A}_\theta \oplus \mathcal{A}_\theta$ and from the definition of the scalar product on $L^2\left(A_{\widetilde{\theta}}, \widetilde{\tau}_0\right)$ it follows that the Condition 5 hold. Condition 6 directly follows from the definition of the operator $\widetilde{D}$. $\square$

7.9. Let us consider a case of $\theta = 0$. In this case $A_{\theta=0}$ is a universal algebra generated by unitary elements $\widehat{u}, \widehat{v}$ with the single relation $\widehat{u}\widehat{v} = \widehat{v}\widehat{u}$ and

$$\mathcal{A}_\theta = \mathcal{A}_0 = \left\{a \in A_0 \mid a = \sum_{r,s} a_{rs}u^r v^s \ \&\ \sup_{r,s\in\mathbb{Z}}\left(1 + r^2 + s^2\right)^k |a_{rs}| < \infty,\ \forall k \in \mathbb{N}\right\} =$$

$$= C^\infty\left(\mathbb{T}^2 = S^1 \times S^1\right)$$

There is a state $\widehat{\tau}_0 : A_0 \to \mathbb{C}$ given by

$$\widehat{\tau}_0\left(\sum_{r,s} a_{rs}\widehat{u}^r\widehat{v}^s\right) = a_{00}.$$

and these is a GNS Hilbert space $L^2\left(A_0, \widehat{\tau}_0\right)$. There are two derivatives $\widehat{\delta}_1$, $\widehat{\delta}_2$ given by

$$\widehat{\delta}_1\left(\sum_{r,s} a_{r,s}\widehat{u}^r\widehat{v}^s\right) = \sum_{rs} 2\pi i r a_{rs}\widehat{u}^r\widehat{v}^s,$$

$$\widehat{\delta}_2\left(\sum_{r,s} a_{r,s}\widehat{u}^r\widehat{v}^s\right) = \sum_{rs} 2\pi i s a_{rs}\widehat{u}^r\widehat{v}^s.$$

There is the natural covering projection $\mathbb{R}^2 \to \mathbb{T}^2$. On can select unbounded functions x, y on $\mathbb{R}^2$ such that $\widehat{u}$ (resp. $\widehat{v}$) corresponds to $e^{2\pi i x}$ (resp $e^{2\pi i y}$) and it is clear that

$$\widehat{\delta}_1 = \frac{\partial}{\partial x},$$

$$\widehat{\delta}_2 = \frac{\partial}{\partial y},$$

i.e. both $\widehat{\delta}_1$ and $\widehat{\delta}_1$ are first-order differential operators. Similarly to 7.6 one can introduce double GNS Hilbert space $\widehat{H} = L^2(A_0, \widehat{\tau}_0) \oplus L^2(A_0, \widehat{\tau}_0)$ and define derivations

$$\widehat{\partial} = \widehat{\partial}_\tau = \widehat{\delta}_1 + \tau\delta_2;\ (\tau \in \mathbb{C},\ \mathrm{Im}(\tau) \neq 0),$$

$$\widehat{\partial}^\dagger = -\widehat{\delta}_1 - \overline{\tau}\widehat{\delta}_2.$$

$$\widehat{D} = \begin{pmatrix} 0 & \widehat{\partial}^\dagger \\ \underline{\widehat{\partial}} & 0 \end{pmatrix}.$$

There is the isomorphism $L^2\left(A_0, \widehat{\tau}_0\right) \approx L^2\left(A_\theta, \tau_0\right)$ of Hilbert spaces given by

$$\sum_{m\in\mathbb{Z}, n\in\mathbb{Z}} a_{mn} u^m v^n \mapsto \sum_{m\in\mathbb{Z}, n\in\mathbb{Z}} a_{mn}\widehat{u}^m\widehat{v}^n.$$

which naturally induces an isomorphism of direct sums $\varphi : \widehat{H} = L^2\left(A_0, \widehat{\tau}_0\right) \bigoplus L^2\left(A_0, \widehat{\tau}_0\right) \approx H = L^2\left(A_\theta, \tau_0\right) \bigoplus L^2\left(A_\theta, \tau_0\right)$. A direct calculation shows that following conditions hold

1. $(\xi, \eta) = (\varphi(\xi), \varphi(\eta))\ \forall \xi, \eta \in L^2\left(A_\theta, \tau_0\right)$,
2. $\mathrm{Dom}\, D = \varphi\left(\mathrm{Dom}\, \widehat{D}\right)$,
3. $D\varphi(\psi) = \varphi\left(\widehat{D}\psi\right)$.

If $A_\theta \to A_{\theta'}$ is given by

$$u \mapsto u'^m;\ v \mapsto v'^n$$

then there is a Hilbert space $\widetilde{H} = A_{\theta'} \otimes_{A_\theta} H$ and the natural action of $G\left(A_{\theta'}|A_\theta\right) \approx \mathbb{Z}_m \times \mathbb{Z}_n$ on $\widetilde{H}$. Similarly there is a the homomorphism $A_0 \to A'_0$ given by

$$\widehat{u} \mapsto \widehat{u}'^m;\ \widehat{v} \mapsto \widehat{v}'^n$$

and the Hilbert space $\widetilde{\widehat{H}} = A'_0 \otimes_{A_0} \widehat{H}$. There is an isomorphism $\widetilde{\varphi} : \widetilde{H} \approx \widetilde{\widehat{H}}$ such that $\widetilde{\varphi}\left(g\xi\right) = g\widetilde{\varphi}\left(\xi\right)$ for any $\xi \in \widetilde{\widehat{H}}$ and $g \in \mathbb{Z}_m \times \mathbb{Z}_n$.

Remark 7.10. The construction 7.9 means that some properties of Hilbert space and its Dirac operator of the noncommutative torus are the same as a Hilbert space and Dirac operator of the commutative torus.

Lemma 7.11. *In the situation described in 7.7 the coherent sequence* $\left\{\left(\mathcal{A}_{\theta_n}, H_n, D_n\right)\right\}_{n \in \mathbb{N}^0}$ *is local.*

Proof. From 7.9 and the Remark 7.10 it follows that general case can be reduced to commutative one, i.e. $\theta = 0$ and $A_0 = C\left(\mathbb{T}^2\right)$. However $\mathbb{T}^2$ is a Riemannian manifold and Dirac operator is local (differential), so an application of the Lemma 6.22 completes the proof of this lemma. □

Lemma 7.12. *The* $\widetilde{\mathcal{A}}_\theta$ *is a dense subalgebra of* $\widetilde{A}_\theta$.

Proof. From the construction 7.4 it follows that for any $\widetilde{a} \in \widetilde{A}_\theta$ and $\varepsilon > 0$ there is a finite subset $G' \subset \mathbb{Z} \times \mathbb{Z}$ and an operator $\widetilde{a}' \in \widetilde{A}_\theta$ given by

$$\widetilde{a}' = \sum_{g,g' \in G'} \sum_{i,j,i',j' \in \{1,2\}} g\xi_{ij}\rangle a^k_{ijgi'j'g'} \langle g'\eta_{i'j'}$$

such that

$$\left\|\widetilde{a} - \widetilde{a}'\right\| < \frac{\varepsilon}{2}.$$

. The element $a^k_{ijgi'j'g'}$ is given by

$$a^k_{ijgi'j'g'} = \left\langle h_k\left(g\right)\left(e^{n_k}_j\left(v_{n_k}\right) e^{m_k}_i\left(u_{m_k}\right)\right), a_k\left(h_k\left(g'\right)\left(e^{m_k}_{i'}\left(u_{m_k}\right) e^{n_k}_{j'}\left(v_{n_k}\right)\right)\right)\right\rangle_{A_{\theta_k}}$$

where $a_k \in A_{\theta_k}$. Since $\mathcal{A}_{\theta_k}$ is dense subalgebra of A_{θ_k} whence there is $\overline{a}_k \in \mathcal{A}_{\theta_k}$ such that

$$\left\|a_k - \overline{a}_k\right\| < \frac{\varepsilon}{8\left|G'\right|}.$$

Since $e^{m_k}_i, e^{n_k}_j \in C^\infty\left(S^1\right)$ we have $e^{m_k}_i\left(u\right) e^{n_k}_j\left(v\right), e^{n_k}_j\left(v\right) e^{m_k}_i\left(u\right) \in \mathcal{A}_{\theta_k}$ and

$$\overline{a}^k_{ijgi'j'g'} = \left\langle h_k\left(g\right)\left(e^{n_k}_j\left(v_{n_k}\right) e^{m_k}_i\left(u_{m_k}\right)\right), \overline{a}_k\left(h_k\left(g'\right)\left(e^{m_k}_{i'}\left(u_{m_k}\right) e^{n_k}_{j'}\left(v_{n_k}\right)\right)\right)\right\rangle_{A_{\theta_k}} \in \mathcal{A}_\theta,$$

If
$$\widetilde{a}'' = \sum_{g,g'\in G'} \sum_{i,j,i',j'\in\{1,2\}} g\xi_{ij}\rangle \overline{a}^k_{ijgi'j'g'} \langle g'\eta_{i'j'}$$
then
$$\left\|\widetilde{a} - \widetilde{a}''\right\| < \varepsilon \tag{95}$$
and
$$\widetilde{a}'' \in \mathcal{K}^\infty\left(\overline{X}^\infty_{\mathcal{A}_\theta}\right).$$
Otherwise if $\{b_r \in A_{\theta_r}\}_{r\in\mathbb{N}}$ is the sequence such that $b_r = 0$ for $r \le k$ and
$$b_r = \sum_{g,g'\in G^k} \sum_{i,j,i',j'\in\{1,2\}} h_r\left(g\right)\left(e^{n_r}_j\left(v_{n_r}\right)e^{m_r}_i\left(u_{m_r}\right)\right)\overline{a}^k_{ijgi'j'g'}\left(h_r\left(g'\right)\left(e^{m_r}_{i'}\left(u_{m_r}\right)e^{n_r}_{j'}\left(v_{n_r}\right)\right)\right)$$
for $r > k$ then $\lim_{r\to\infty} b_r = \widetilde{a}''$ in the weak topology i.e. $\widetilde{a}'' \in \left(\bigcup A_{\theta_k}\right)''$, whence $\widetilde{a}'' \in \widetilde{\mathcal{A}}_\theta$. From (95) it follows that $\widetilde{\mathcal{A}}_\theta$ is dense in $\widetilde{A}_\theta$.

□

From the Lemma 7.12 it follows that the described in 7.7 sequence of spectral triples is regular.

8 Epilogue

There is a good noncommutative generalization of covering projections [18] based on monads, comonads and adjoint functors. But this generalization cannot describe coverings of locally compact spaces. In the Definition 4.4 finite sums and projective modules are *manually* replaced with infinite sums and compact operators. In result we have no a beautiful theory, but we have new constructions. This replacement can be regarded as illegal. However Max Planck *manually* replaced integrals with sums [32] and the quantum mechanics had been obtained.

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